COMPLETE $M$ CONVEX ALGEBRAS WHOSE POSITIVE ELEMENTS ARE TOTALLY ORDERED
A. EL KINANI, M.A. NEJJARI & M. OUDADESS

Abstract: We show that unitary and complete $l. m. c. a.'s$ endowed with certain orders are actually locally $C^\ast$-algebras or even reduce to the complex field.

Keywords: Positive elements, $l. m. c. a.$, locally $C^\ast$-algebra.

1. Introduction

The aim of this note is to extend to locally $m$-convex algebras the results of [3]. The matter is then to study the structure of unitary and complete $l. m. c. a.'s$ whose positive elements are totally ordered; and this relatively to the orders defined by the cones $A_+ = \{x \in \text{Sym}(A) : Spx \subset R_+\}$ and $P = \{x \in A : V(x) \subset R_+\}$. In a locally $C^\ast$-algebra (which is of course hermitian), we always have $A_+ = P$. As a converse, we show that, in a complex unital hermitian and complete $m$-convex algebra, if $A_+ \subset P$, then it is a locally $C^\ast$-algebra (Theorem 3.1). It is also known that in a locally $C^\ast$-algebra, the cone of positive elements is partially ordered and $A_+ = P$. One may ask whether or not it can be totally ordered. In fact, the last condition appears to be restrictive as propositions 3.2 and 3.4 show.

2. Preliminaries

Let $(A, (\| \cdot \|)_\lambda)$ be a complex unitary and complete locally $m$-convex algebra ($l.m.c.a.$ in short). It is known that $(A, (\| \cdot \|)_\lambda)$ is the projective limit of the normed algebras $(A_\lambda, \| \cdot \|_\lambda)$, where $A_\lambda = A/N_\lambda$ with $N_\lambda = \{x \in A : \|x\|_\lambda = 0\}$, and $\|x\|_\lambda = \|x\|_\lambda$, $x_\lambda \equiv x + N_\lambda$. An element $x$ of $A$ is written $x = (x_\lambda)_\lambda = (\pi_\lambda(x))_\lambda$, where $\pi_\lambda : A \to A_\lambda$ is the canonical surjection. The algebra $(A, (\| \cdot \|)_\lambda)$ is also the projective limit of the Banach algebras $A_\lambda$, the completions of $A_\lambda$'s. The norm in $A_\lambda$ will also be denoted by $\| \cdot \|_\lambda$. The numerical range of an element

2000 Mathematics Subject Classification: 46K99, 46H20.
\( a \in A \) is denoted by \( V(a) \). Recall that \( V(a) = \bigcup_{\lambda} V(\hat{A}_\lambda, a_\lambda) \), where \( V(\hat{A}_\lambda, a_\lambda) \) is the numerical range of \( a_\lambda \) in the Banach algebra \( \hat{A}_\lambda \). We consider the subsets \( P = \{ x \in A : V(x) \subset R_+ \} \) and \( H = \{ x \in A : V(x) \subset R \} \). The first subset is said to be the cone of positive elements, of \( A \), relatively to the numerical range. Let \( (A, \langle |.| \rangle_\lambda) \) be a l.m.c.a. endowed with an involution \( x \mapsto x^* \). The set of all hermitian elements (i.e., all \( x \) such that \( x = x^* \)) will be denoted by \( \text{Sym}(A) \). We say that the algebra \( A \) is hermitian if the spectrum of every element of \( \text{Sym}(A) \) is real ([2]). It is said to be symmetric if \( e + xx^* \) is invertible, for every \( x \) in \( A \). Put \( A_+ = \{ x \in \text{Sym}(A) : Sp(x) \subset R_+ \} \), the set of all positive elements, of \( A \), relatively to the involution. If \( A \) is symmetric then \( A_+ \) is a convex cone. A locally \( C^* \)-algebra ([4]) is a complete l.m.c.a. \( (A, \langle |.| \rangle_\lambda) \) endowed with an involution \( x \mapsto x^* \) such that, for every \( \lambda \), \( |x^*|_{\lambda} = |x|_{\lambda}^* \), for every \( x \in A \). Concerning involutive l. m. c. a.'s, the reader is referred to [2]. In the sequel, all algebras are complex. The spectral radius will be denoted by \( \rho \) that is \( \rho(x) = \sup \{|z| : z \in Sp(x)\} \), where \( Sp(x) \) is the spectrum of \( x \).

### 3. Structure results

It is not always true that \( A_+ \subset P \) as the following result shows.

**Theorem 3.1.** Let \( (A, \langle |.| \rangle_\lambda) \) be an involutive commutative, unitary, complete and hermitian l.m.c.a. If \( A_+ \subset P \), then \( A \) is a locally \( C^* \)-algebra for an equivalent family of semi-norms.

**Proof.** Since the algebra is hermitian, we have \( \text{Sym}(A) = A_+ - A_- \) for \( h = (h^2 + e) - (h^2 - h + e) \), for every \( h \in \text{Sym}(A) \). On the other hand, \( A_+ \) satisfies the following condition

\[
(e + u)^{-1} \in A_+ \text{ for every } u \in A_+.
\]

(1)

Now \( P_\lambda = \pi_\lambda(P) \subset \hat{P}_\lambda \) where \( \hat{P}_\lambda = \{ a \in \hat{A}_\lambda : V(\hat{A}_\lambda, a) \subset R_+ \} \). But \( \hat{P}_\lambda \) is normal; whence the normality of \( P \) follows and so the one of \( A_+ \). The convex cone \( \pi_\lambda(A_+) \), in \( \hat{A}_\lambda \), is stable by product, normal and satisfies (1). By ([1], proposition 12, p. 258), we have \( \pi_\lambda(A_+) \subset \{ u \in \hat{A}_\lambda : Spu \subset R_+ \} \). The closed convex cone \( K_\lambda = \pi_\lambda(A_+) \) satisfies also these properties. Put \( B_\lambda = K_\lambda - K_\lambda \), a real subalgebra, of \( \hat{A}_\lambda \), generated by \( K_\lambda \). It is closed by ([1], theorem 2, p. 260). We now show that the complex subalgebra \( B_\lambda + iB_\lambda \) is closed in \( \hat{A}_\lambda \). Using the normality of \( K_\lambda \), one obtains that, for every \( \lambda \), there is \( \beta_\lambda > 0 \) such that, for every \( h \in B_\lambda \), \( \|h\|_\lambda \leq \beta_\lambda \|h + ik\|_\lambda \), for every \( k \in B_\lambda \). Whence the closedness of \( B_\lambda + iB_\lambda \). But \( A_\lambda = \pi_\lambda(A) \subset B_\lambda + iB_\lambda \). Hence \( B_\lambda + iB_\lambda \) is dense in \( \hat{A}_\lambda \). Whence \( B_\lambda + iB_\lambda \subset \hat{A}_\lambda \). By ([1], theorem 2, p. 260), we have \( Sp(h) \subset R \), for every \( h \in B_\lambda \). Moreover \( B_\lambda \cap iB_\lambda = \{0\} \), due to the normality of \( K_\lambda \). Hence a hermitian involution \( (h + ik)^* = (h - ik) \).
$h-ik$, is defined on $\widehat{A}_\lambda$. At last, again the normality of $K_\lambda$ implies $\|h\|_\lambda \leq \alpha \varrho(h)$, for some $\alpha > 0$ and every $h$ in $B_\lambda$. We conclude by a result of Ptàk ([6]; (8.4) Theorem).

If the order is total, we do not need the commutativity and the conclusions show that this condition is very strong.

We begin with the order associated to $A_+$. 

**Proposition 3.2.** Let $(A, (|.|)_\lambda)$ be an involutive, unitary and complete l. m. c. a. If $(A_+, \leq)$ is totally ordered, then $A_+ = R_+$. 

**Proof.** We first show that $g(x) < +\infty$, for every $x \in A_+$. Since the order is total on $A_+$, we have $x \leq n$ or $n \leq x$, for every $n \in N^*$. If $Sp x$ is unbounded, then $n \leq x$, for every $n$, a contradiction with $Sp x \not\subseteq \emptyset$ ([5]). Suppose now that $x \in A_+$ and $0 \in Sp x$. For every $\alpha > 0$, one gets $x \leq \alpha$, for otherwise $x < 0$. Whence $Sp x = \{0\}$ and hence $x = 0$. On the other hand, if $x \in A_+$ and $0 \not\in Sp x$, put $m = \inf \{\beta : \beta \in Sp x\}$. Then one has $0 \in Sp (x - m)$ otherwise $x - m$ would be invertible and $g((x - m)^{-1}) = +\infty$; a contradiction for $(x - m)^{-1} \in A_+$. Hence $x = m \in A_+$. 

An interesting application of this proposition is contained in the following result.

**Corollary 3.3.** Let $(A, (|.|)_\lambda)$ be an involutive, unitary and complete l. m. c. a. If $(A_+, \leq)$ is totally ordered, then

(i) $\{x \in \text{Sym}(A) : Sp x \subseteq R\} = R$,

(ii) If $A$ is hermitian, then $A = C$.

**Proof.** (i) Every $x \in \text{Sym}(A)$ with $Sp x \subseteq R$ can be written $x = (x^2 - c) - (x^2 - x + e)$. And then the assertion (ii) follows immediately from (i).

We now examine the order associated to $P$

**Proposition 3.4.** Let $(A, (|.|)_\lambda)$ be a unitary and complete l. m. c. a. If $(P, \leq)$ is totally ordered, then $P = R_+$. 

**Proof.** Let $x \in P$ and $r = \inf \{\alpha : \alpha \in V(x)\}$. Then, for every $n \in N^*$, we have $x \leq r + \frac{1}{n}$; otherwise there is $n_0 \in N^*$ such that $r + \frac{1}{n_0} < x$, i.e. $V(x - r - \frac{1}{n_0}) \subseteq R_+$. Due to the definition of $V(x - r - \frac{1}{n})$, one immediately checks that $r + \frac{1}{n_0} < \alpha$, for every $\alpha$ in $V(x)$. Hence $r + \frac{1}{n_0} \leq r$; a contradiction. Now $x \leq r + \frac{1}{n}$ means $\beta \leq r + \frac{1}{n}$, for every $\beta$ in $V(x)$. So $V(x) \subseteq [r, r + \frac{1}{n}]$, for every $n$. And since $V(x)$ is non void, we get $V(x) = \{\beta_0\}$. Whence $x = \beta_0$.

We have the following consequence.

**Corollary 3.5.** Let $(A, (|.|)_\lambda)$ be a unitary and complete l. m. c. a. If $(P, \leq)$ is totally ordered and $A = H + iH$, then $A$ is isomorphic to $C$.

**Proof.** Since every $h \in H$ can be written $h = \frac{1}{2} \left[(h + e)^2 - (h^2 + e)\right]$, it is sufficient to show that $h^2 \in P$, for every $h \in H$. Let $p, q \in H$ such that
\[ h^2 = p + iq. \] We have \[ h^2 = p_\lambda + iq_\lambda \] in \( \overline{A}_\lambda \) for every \( \lambda \), with \( p_\lambda, q_\lambda \in H_\lambda \), where \( H_\lambda = \{ u \in A_\lambda : V(A_\lambda, u) \subset R \} \). The identity \( h_\lambda h^2 = h^2 h_\lambda \) yields \( h_\lambda p_\lambda - p_\lambda h_\lambda = i(q_\lambda h_\lambda - h_\lambda q_\lambda) \). Whence \( h_\lambda p_\lambda - p_\lambda h_\lambda \in H_\lambda \cap iH_\lambda \) ([1], lemma 2, p. 206). Hence \( h_\lambda p_\lambda = p_\lambda h_\lambda \); and so \( p_\lambda q_\lambda = q_\lambda p_\lambda \). We then have \( V(h^3) \subset Co(Sphh^3) \subset R_+ \), where \( Co \) stands for the convex hull. The first inclusion is due to [1], lemma 4, p. 206. It follows that \( V(h^2) \subset R_+ \). □

References


Address: Ecole Normale Supérieure, B.P.5118-Takaddoum, 10105 Rabat, Maroc
Received: 15 May 2001