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COMPLETE M-CONVEX ALGEBRAS WHOSE POSITIVE ELEMENTS ARE TOTALLY ORDERED

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Abstract: We show that unitary and complete $l.\ m.\ c.\ a.$'s endowed with certain orders are actually locally C^*-algebras's or even reduce to the complex field.
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1. Introduction

The aim of this note is to extend to locally m-convex algebras the results of [3]. The matter is then to study the structure of unitary and complete $l.\ m.$ c. $a.$'s whose positive elements are totally ordered; and this relatively to the orders defined by the cones $A_+ = \{x \in Sym(A) : Spx \subset R_+\}$ and $P = \{x \in A : V(x) \subset R_+\}$. In a locally C^*-algebra (which is of course hermitian), we always have $A_+ = P$. As a converse, we show that, in a complex unital hermitian and complete m-convex algebra, if $A_+ \subset P$, then it is a locally C^*-algebra (Theorem 3.1). It is also known that in a locally C^*-algebra, the cone of positive elements is partially ordered and $A_+ = P$. One may ask whether or not it can be totally ordered. In fact, the last condition appears to be restrictive as propositions 3.2 and 3.4 show.

2. Preliminaries

Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a complex unitary and complete locally m-convex algebra ($l.m.c.a.$ in short). It is known that $\left(A, (|.|_\lambda)_\lambda\right)$ is the projective limit of the normed algebras $(A_\lambda, \|.\|_\lambda)$, where $A_\lambda = A/N_\lambda$ with $N_\lambda = \{x \in A : |x|_\lambda = 0\}$; and $\|x_\lambda\|_\lambda = |x|_\lambda$, $x_\lambda \equiv x + N_\lambda$. An element x of A is written $x = (x_\lambda)_\lambda = (\pi_\lambda(x))_\lambda$, where $\pi_\lambda : A \longrightarrow A_\lambda$ is the canonical surjection. The algebra $\left(A, (|.|_\lambda)_\lambda\right)$ is also the projective limit of the Banach algebras $\widehat{A_\lambda}$, the completions of A_λ's. The norm in $\widehat{A_\lambda}$ will also be denoted by $\|.\|_\lambda$. The numerical range of an element

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$a \in A$ is denoted by $V(a)$. Recall that $V(a) = \bigcup_\lambda V(\widehat{A_\lambda}, a_\lambda)$, where $V(\widehat{A_\lambda}, a_\lambda)$ is the numerical range of a_λ in the Banach algebra $\widehat{A_\lambda}$. We consider the subsets $P = \{x \in A : V(x) \subset R_+\}$ and $H = \{x \in A : V(x) \subset R\}$. The first subset is said to be the cone of positive elements, of A, relatively to the numerical range. Let $(A, (|.|_\lambda)_\lambda)$ be a *l.m.c.a.* endowed with an involution $x \longmapsto x^*$. The set of all hermitian elements (i.e., all x such that $x = x^*$) will be denoted by $Sym(A)$. We say that the algebra A is hermitian if the spectrum of every element of $Sym(A)$ is real ([2]). It is said to be symmetric if $e + xx^*$ is invertible, for every x in A. Put $A_+ = \{x \in Sym(A),\ Spx \subset R_+\}$, the set of all positive elements, of A, relatively to the involution. If A is symmetric then A_+ is a convex cone. A locally C^*-algebra ([4]) is a complete *l.m.c.a.* $(A, (|.|_\lambda)_\lambda)$ endowed with an involution $x \longmapsto x^*$ such that, for every λ, $|x^*x|_\lambda = |x|_\lambda^2$, for every $x \in A$. Concerning involutive *l. m. c. a.* 's, the reader is refered to [2]. In the sequel, all algebras are complex. The spectral radius will be denoted by ϱ that is $\varrho(x) = \sup\{|z| : z \in Spx\}$, where Spx is the spectrum of x.

3. Structure results

It is not always true that $A_+ \subset P$ as the following result shows.

Theorem 3.1. *Let $(A, (|.|_\lambda)_\lambda)$ be an involutive commutative, unitary, complete and hermitian l. m. c. a. If $A_+ \subset P$, then A is a locally C^*-algebra for an equivalent family of semi-norms.*

Proof. Since the algebra is hermitian, we have $Sym(A) = A_+ - A_+$ for $h = (h^2 + e) - (h^2 - h + e)$, for every $h \in Sym(A)$. On the other hand, A_+ satisfies the following condition

$$(\mathrm{e} + u)^{-1} \in A_+; \text{ for every } u \in A_+. \tag{1}$$

Now $P_\lambda = \pi_\lambda(P) \subset \widehat{P}_\lambda$ where $\widehat{P}_\lambda = \left\{a \in \widehat{A_\lambda} : V(\widehat{A_\lambda}, a) \subset R_+\right\}$. But $\widehat{P}_\lambda$ is normal; whence the normality of P follows and so the one of A_+. The convex cone $\pi_\lambda(A_+)$, in $\widehat{A_\lambda}$, is stable by product, normal and satisfies (1). By ([1], proposition 12, p. 258), we have $\pi_\lambda(A_+) \subset \left\{u \in \widehat{A_\lambda} : Spu \subset R_+\right\}$. The closed convex cone $K_\lambda = \overline{\pi_\lambda(A_+)}$ satisfies also these properties. Put $B_\lambda = K_\lambda - K_\lambda$, a real subalgebra, of $\widehat{A_\lambda}$, generated by K_λ. It is closed by ([1], theorem 2, p. 260). We now show that the complex subalgebra $B_\lambda + iB_\lambda$ is closed in $\widehat{A_\lambda}$. Using the normality of K_λ, one obtains that, for every λ, there is $\beta_\lambda > 0$ such that, for every $h \in B_\lambda$, $\|h\|_\lambda \leq \beta_\lambda \|h + ik\|_\lambda$, for every $k \in B_\lambda$. Whence the closedness of $B_\lambda + iB_\lambda$. But $A_\lambda = \pi_\lambda(A) \subset B_\lambda + iB_\lambda$. Hence $B_\lambda + iB_\lambda$ is dense in $\widehat{A_\lambda}$. Whence $B_\lambda + iB_\lambda = \widehat{A_\lambda}$. By ([1], theorem 2, p. 260), we have $Sph \subset R$, for every $h \in B_\lambda$. Moreover $B_\lambda \cap iB_\lambda = \{0\}$, due to the normality of K_λ. Hence a hermitian involution $(h + ik)^* =$

$h-ik$, is defined on $\widehat{A_\lambda}$. At last, again the normality of K_λ implies $\|h\|_\lambda \leq \alpha\varrho(h)$, for some $\alpha > 0$ and every h in B_λ. We conclude by a result of Ptàk ([6]; (8,4) Theorem). ■

If the order is total, we do not need the commutativity and the conclusions show that this condition is very strong.

We begin with the order associated to A_+.

Proposition 3.2. *Let $(A, (|.|_\lambda)_\lambda)$ be an involutive, unitary and complete l. m. c. a. If $(A_+, \leq)$ is totally ordered, then $A_+ = R_+$.*

Proof. We first show that $\varrho(x) < +\infty$, for every $x \in A_+$. Since the order is total on A_+, we have $x \leq n$ or $n \leq x$, for every $n \in N^*$. If Spx is unbounded, then $n \leq x$, for every n; a contradiction with $Spx \neq \emptyset$ ([5]). Suppose now that $x \in A_+$ and $0 \in Spx$. For every $\alpha > 0$, one gets $x \leq \alpha$, for otherwise $\alpha < 0$. Whence $Spx = \{0\}$ and hence $x = 0$. On the other hand, if $x \in A_+$ and $0 \notin Spx$, put $m = \inf\{\beta : \beta \in Spx\}$. Then one has $0 \in Sp(x-m)$ otherwise $x - m$ would be invertible and $\varrho\left((x-m)^{-1}\right) = +\infty$; a contradiction for $(x-m)^{-1} \in A_+$. Hence $x = m \in A_+$. ■

An interesting application of this proposition is contained in the following result.

Corollary 3.3. *Let $(A, (|.|_\lambda)_\lambda)$ be an involutive, unitary and complete l. m. c. a. If $(A_+, \leq)$ is totally ordered, then*

(i) $\{x \in Sym(A) : Spx \subset R\} = R$,

(ii) If A is hermitian, then $A = C$.

Proof. (i) Every $x \in Sym(A)$ with $Spx \subset R$ can be written $x = (x^2+e) - (x^2 - x + e)$. And then the assertion (ii) follows immediately from (i). ■

We now examine the order associated to P.

Proposition 3.4. *Let $(A, (|.|_\lambda)_\lambda)$ be a unitary and complete l. m. c. a. If $(P, \leq)$ is totally ordered, then $P = R_+$.*

Proof. Let $x \in P$ and $r = \inf\{\alpha : \alpha \in V(x)\}$. Then, for every $n \in N^*$, we have $x \leq r + \frac{1}{n}$; otherwise there is $n_0 \in N^*$ such that $r + \frac{1}{n_0} < x$, i.e. $V(x - r - \frac{1}{n_0}) \subset R_+$. Due to the definition of $V(x - r - \frac{1}{n_0})$, one immediately checks that $r + \frac{1}{n_0} < \alpha$, for every α in $V(x)$. Hence $r + \frac{1}{n_0} \leq r$; a contradiction. Now $x \leq r + \frac{1}{n}$ means $\beta \leq r + \frac{1}{n}$, for every β in $V(x)$. So $V(x) \subset \left[r,\ r + \frac{1}{n}\right]$, for every n. And since $V(x)$ is non void, we get $V(x) = \{\beta_0\}$. Whence $x = \beta_0$. ■

We have the following consequence.

Corollary 3.5. *Let $(A, (|.|_\lambda)_\lambda)$ be a unitary and complete l. m. c. a. If $(P, \leq)$ is totally ordered and $A = H + iH$, then A is isomorphic to C.*

Proof. Since every $h \in H$ can be written $h = \frac{1}{2}\left[(h+e)^2 - (h^2 + e)\right]$, it is sufficient to show that $h^2 \in P$, for every $h \in H$. Let $p, q \in H$ such that

$h^2 = p + iq$. We have $h_\lambda^2 = p_\lambda + iq_\lambda$ in $\widehat{A_\lambda}$ for every λ, with $p_\lambda, q_\lambda \in H_\lambda$, where $H_\lambda = \left\{u \in \widehat{A_\lambda} : V(\widehat{A_\lambda}, u) \subset R\right\}$. The identity $h_\lambda h_\lambda^2 = h_\lambda^2 h_\lambda$ yields $h_\lambda p_\lambda - p_\lambda h_\lambda = i(q_\lambda h_\lambda - h_\lambda q_\lambda)$. Whence $h_\lambda p_\lambda - p_\lambda h_\lambda \in H_\lambda \cap iH_\lambda$ ([1], lemma 2, p. 206). Hence $h_\lambda p_\lambda = p_\lambda h_\lambda$; and so $p_\lambda q_\lambda = q_\lambda p_\lambda$. We then have $V(h_\lambda^2) \subset Co(Sph_\lambda^2) \subset R_+$, where Co stands for the convex hull. The first inclusion is due to [1], lemma 4, p. 206. It follows that $V(h^2) \subset R_+$. ■

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