ON THE INTEGRAL OF THE ERROR TERM IN THE FOURTH MOMENT OF THE RIEMANN ZETA-FUNCTION

ALEKSANDAR IVIĆ

1. Introduction

The aim of this note is to provide an asymptotic formula for \( \int_0^T E_2(t) \, dt \), where \( E_2(T) \) is the error term in the asymptotic formula for the fourth moment of \( |\zeta(\frac{1}{2} + it)| \). The asymptotic formula for the fourth moment of the Riemann zeta-function \( \zeta(s) \) on the critical line is customarily written as

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = TP_4(\log T) + E_2(T),
\]

where

\[
P_4(x) = \sum_{j=0}^{4} a_j x^j.
\]

It is classically known that \( a_4 = 1/(2\pi^2) \), and it was proved by D. R. Heath-Brown [1] that

\[
a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta''(2)\pi^{-2})\pi^{-2}.
\]

He also produced more complicated expressions for \( a_0, a_1 \) and \( a_2 \) in (1.2) (\( \gamma = 0.577\ldots \) is Euler’s constant). For an explicit evaluation of the \( a_j \)’s the reader is referred to [4].

In recent years, due primarily to the application of powerful methods of spectral theory (see Y. Motohashi’s monograph [13] for a comprehensive account), much advance has been made in connection with \( E_2(T) \). We refer the reader to the works [5]–[9], [11]–[13] and [16]. It is known now that

\[
E_2(T) = O(T^{2/3} \log C_1 T), \quad E_2(T) = O(T^{1/2}),
\]

\[
\int_0^T E_2(t) \, dt = O(T^{3/2}), \quad \int_0^T E_2^2(t) \, dt = O(T^2 \log C_2 T).
\]
with effective constants $C_1, C_2 > 0$ (the values $C_1 = 8, C_2 = 22$ are worked out in [13]). The above results were proved by Y. Motohashi and the author: (1.3) and the first bound in (1.4) in [3], [8], [13] and the second upper bound in (1.4) in [7]. The omega-result in (1.3) $f = O(g)$ means that $f = o(g)$ does not hold, $f = \Omega_{+}(g)$ means that \( \limsup f/g > 0 \) and that \( \liminf f/g < 0 \) was improved to $E_2(T) = \Omega_T(T^{1/2})$ by Y. Motohashi [12]. Recently the author [6] made further progress in this problem by proving the following quantitative omega-result: there exist two constants $A > 0, B > 1$ such that for $T \geq T_0 > 0$ every interval $[T, BT]$ contains points $T_1, T_2$ for which

$$ E_2(T_1) > AT_1^{1/2}, \quad E_2(T_2) < -AT_2^{1/2}. \quad (1.5) $$

There is an obvious discrepancy between the $O$–result and $\Omega$–result in (1.3), and it may be well conjectured that $E_2(T) = O_{\varepsilon}(T^{1/2+\varepsilon})$ for any given $\varepsilon > 0$ ($\varepsilon$ will denote arbitrarily small constants, not necessarily the same ones at each occurrence). This bound, if true, is very strong, since it would imply (e.g., by Lemma 7.1 of [3]) the hitherto unproved bound $\zeta'(1/2 + it) \ll t^\varepsilon \log^{1/2} t$. The upper bound in (1.3) seems to be the limit of the existing methods, since the only way to estimate the relevant exponential sum in this problem, namely (see [3], [8] and [13])

$$ \sum_{K < \kappa_j \leq 2K} \alpha_j H_j^3(\frac{1}{2}) \exp \left( ik_j \log \left( \frac{T}{\kappa_j} \right) \right) \quad (1 \ll K \leq T^{1/2}) \quad (1.6) $$

appears to be trivial estimation, coming from the bound

$$ \sum_{K < \kappa_j \leq 2K} \alpha_j \left| H_j^2(\frac{1}{2}) \right| \ll K^2 \log^C K \quad (C > 0). \quad (1.7) $$

This follows by the Cauchy-Schwarz inequality from the bounds (see [13])

$$ \sum_{\kappa_j \leq K} \alpha_j H_j^2(\frac{1}{2}) \ll K^2 \log K, \quad \sum_{\kappa_j \leq K} \alpha_j H_j^4(\frac{1}{2}) \ll K^2 \log^{15} K \quad (1.8) $$

with $C = 8$ in (1.7). Here as usual \{ $\lambda_j = \kappa_j^2 + \frac{1}{4} \} \cup \{ 0 \}$ denotes the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2, \mathbb{Z})$–automorphic forms, and $\alpha_j = | \rho_j(1) |^2 (\cosh \pi \kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue $\lambda_j$ to which the Hecke series $H_j(s)$ is attached. It is precisely the presence of $H_j^2(\frac{1}{2})$ in (1.6) which makes the sum in question very hard to deal with, and any decrease of the exponent 2/3 in the upper bound for $E_2(T)$ in (1.3) will likely involve the application of genuine new ideas.

In [6] the author proved that there exist constants $A > 0$ and $B > 1$ such that, for $T \geq T_0 > 0$, every interval $[T, BT]$ contains points $t_1, t_2$ for which

$$ \int_0^{t_1} E_2(t) \, dt > At_1^{3/2}, \quad \int_0^{t_2} E_2(t) \, dt < -At_2^{3/2}. \quad (1.9) $$
This result, of course, implies that \( \int_0^T E_2(t) \, dt = M_T(T^{3/2}) \). It was also used in [6] to prove a lower bound result, whose special case \( a = 2 \) gives

\[
\int_0^T E_2^2(t) \, dt \gg T^2. \tag{1.10}
\]

Thus sharpening (1.8) and showing that the upper bound in (1.4) is very close to the true order of magnitude of the mean square integral of \( E_2(T) \).

The main aim of this paper is to prove a result, which gives an asymptotic formula for the integral of \( E_2(t) \), thereby sharpening the first bound in (1.4). This is the following

**Theorem 1.1.** Let

\[
\eta(T) := (\log T)^{3/5} (\log \log T)^{-1/5}. \tag{1.11}
\]

\[
R_1(\kappa_h) := \sqrt{\frac{\pi}{2}} \left( 2^{-i\kappa_h} \frac{\Gamma(\frac{1}{4} - \frac{1}{2}i\kappa_h)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\kappa_h)} \right)^3 \Gamma(2i\kappa_h) \cosh(\pi\kappa_h). \tag{1.12}
\]

Then there exists a constant \( C > 0 \) such that

\[
\int_0^T E_2(t) \, dt = 2T^{\frac{3}{2}} \Re \left\{ \sum_{j=1}^N \alpha_j H_j^3(\frac{1}{2}) \left( \frac{T^{1/2}}{(\frac{1}{2} + i\kappa_j)(\frac{3}{2} + i\kappa_j)} \right) R_1(\kappa_j) \right\} + O(T^{\frac{3}{2}} e^{-C\eta(T)}). \tag{1.13}
\]

From Stirling’s formula for the gamma-function it follows that \( R_1(\kappa_j) \ll \kappa_j^{-1/2} \), hence by (1.7) and partial summation it follows that the series on the right-hand side of (1.13) is absolutely convergent, and it can be also shown (see [3], [5], [6]) that \( \Re \{ \ldots \} \) is also \( \Omega \pm 1 \). Thus from Theorem 1.1 we can easily deduce all previously known \( \Omega \)-results for \( E_2(T) \). The error term in (1.13) is similar to the error term in the strongest known form of the prime number theorem (see e.g., [2, Chapter 12]). This is by no means a coincidence, and the reason for such a shape of the error term in (1.13) will transpire from the proof of Theorem 1.1, which will be given in Section 3.

**2. A mean square result**

We shall deduce the proof of Theorem 1.1 from a mean square result for the function

\[
Z_2(s) := \int_1^\infty |\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx \quad (\Re s = \sigma > 1). \tag{2.1}
\]

It was introduced and studied in [12], [13, Chapter 5], and then further used and studied in [5], [6] and [9]. Y. Motohashi [12] has shown that \( Z_2(s) \) has meromorphic continuation over \( \mathbb{C} \). In the half-plane \( \Re s > 0 \) it has the following
singularities: the pole $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_{j}$ ($\kappa_{j} = \sqrt{\lambda_{j}} - 1/4$) and poles at $s = \frac{1}{2} \rho$, where $\rho$ denotes complex zeros of $\zeta(s)$. The residue of $Z_{2}(s)$ at $s = \frac{1}{2} + i\kappa_{h}$ equals

$$R(\kappa_{h}) := \sqrt{\frac{\pi}{2}} \left( 2^{-i\kappa_{h}} \frac{\Gamma(\frac{1}{4} - \frac{3}{2} \mathbf{i} \kappa_{h})}{\Gamma(\frac{1}{4} + \frac{3}{2} \mathbf{i} \kappa_{h})} \right)^{3} \Gamma(2i\kappa_{h}) \cosh(\pi \kappa_{h}) \sum_{\kappa_{j} = \kappa_{h}} \alpha_{j} H_{j}^{3}(\frac{1}{2}),$$

and the residue at $s = \frac{1}{2} - i\kappa_{h}$ equals $\bar{R}(\kappa_{h})$. The function $Z_{2}(s)$ is a natural tool for investigations involving $E_{2}(T)$ (see (3.3) and (3.4)). Its spectral decomposition (see [12] and [13, Chapter 5]) enables one to connect problems with $E_{2}(T)$ to results from spectral theory. We shall prove the following

**Theorem 2.1.** Let

$$\sigma = \frac{1}{2} - C \delta(V), \quad \delta(V) := (\log V)^{-2/3} (\log \log V)^{-1/3},$$

(2.2)

where $C > 0$ is a suitable constant. Then

$$\int_{V}^{2V} |Z_{2}(\sigma + it)|^{2} dt \ll_{\varepsilon} V^{2 + \varepsilon}.$$  

(2.3)

**Proof.** We note that in [9] the bound (2.3) was shown to hold for $\frac{1}{2} < \sigma < 1$, but it is the region $\sigma < \frac{1}{2}$ that is more difficult to deal with. As in [9] we write

$$Z_{2}(s) = \int_{1}^{\infty} I(T, \Delta) T^{-s} dT + \int_{1}^{\infty} (|\zeta(\frac{1}{2} + iT)|^{4} - I(T, \Delta)) T^{-s} dT$$

(2.4)

$$= Z_{21}(s) + Z_{22}(s),$$

say, where

$$I(T, \Delta) = \frac{1}{\sqrt{\pi \Delta}} \int_{-\infty}^{\infty} \zeta(\frac{1}{2} + iT + t))^{4} \exp\left(-\frac{t^{2}}{\Delta}\right) dt \quad (\Delta = T^{\xi}, \frac{1}{3} \leq \xi \leq \frac{1}{2}).$$

(2.5)

Before we pass to specific bounds, we shall discuss the method that will be used. Let us suppose that we want to obtain an upper bound for

$$I := \int_{T}^{2T} \left| \int_{a}^{b} g(x)^{-s} dx \right|^{2} dt \quad (s = \sigma + it, T \geq T_{0} > 0),$$

(2.6)

where $g(x)$ is a real-valued, integrable function on $[a, b]$, a subinterval of $[1, \infty)$ (which is not necessarily finite), and which satisfies $g(x) \ll x^{C}$ for some $C > 0$. Let $\varphi(x) \in C_{0}^{\infty}(0, \infty)$ be a test function such that $\varphi(x) \geq 0$, $\varphi(x) - 1$ for
\[ T \leq x \leq 2T, \quad \varphi(x) = 0 \text{ for } x < \frac{1}{2}T \text{ or } x > \frac{5}{2}T \quad (T \geq 10 > 0), \quad \varphi(x) \text{ is increasing in } [\frac{1}{2}T, T] \text{ and decreasing in } [2T, \frac{5}{2}T]. \]
Then we have, by \( r \) integrations by parts,

\[ \int_{T/2}^{5T/2} \varphi(t) \left( \frac{y}{x} \right)^{i} dt = (-1)^{r} \int_{T/2}^{5T/2} \varphi^{(r)}(t) \left( \frac{y}{x} \right)^{i} \left( \frac{i \log(y/x)}{\log \frac{y}{x}} \right)^{r} dt \]

\[ \ll_{r} r^{i} \cdot \left| \log \frac{y}{x} \right|^{-r} \ll T^{-\alpha} \quad (2.7) \]

for any fixed \( A > 0 \) and any given \( \varepsilon > 0 \), provided that \(|y - r| \geq rT^{\varepsilon-1} \) and \( r = r(A, \varepsilon) \) is large enough. Recalling that \( g(x) \ll x^{C} \) and using (2.7) it follows that

\[ I \leq \int_{T/2}^{5T/2} \varphi(t) \left( \int_{a}^{b} g(x)x^{-s} \, dx \right)^{2} \left( \frac{y}{x} \right)^{i} dt \]

\[ = \int_{a}^{b} g(x)g(y)(xy)^{-\sigma} \int_{T/2}^{5T/2} \varphi(t) \left( \frac{y}{x} \right)^{i} \, dt \, dx \, dy \quad (2.8) \]

\[ \ll 1 + \int_{T/2}^{5T/2} \varphi(t) \int_{a}^{b} |g(x)|x^{-\sigma} \int_{-x^{-T^{\varepsilon-1}}}^{-x^{-T^{\varepsilon-1}}} |g(y)|y^{-\sigma} \, dy \, dx \, dt. \]

and the problem is reduced to the estimation of the integral of \( g(x) \) over short intervals: here actually \( g(x) \) does not have to be real-valued. In (2.8) we may further use the elementary inequality \(|g(x)g(y)| \leq \frac{1}{2}(g^{2}(x) + g^{2}(y))\). and thus reduce the problem to mean square estimates.

In the expression for \( Z_{22}(s) \) in (2.4) we denote by \( I_{1}(s, X) \) the integral in which \( T \leq X \), and by \( I_{2}(s, X) \) the remaining integral, where \( X (\ll V^{C}) \) is a parameter to be chosen later. We have (\( s = \sigma + it \))

\[ \int_{V}^{2V} |I_{1}(s, X)|^{2} dt \ll \int_{V}^{2V} \left| \int_{1}^{X} |\zeta(\frac{1}{2} + iT)|^{4}T^{-s} \, dT \right|^{2} dt \]

\[ + \int_{V}^{2V} \left| \int_{-\log V}^{\log V} \int_{1}^{X} |\zeta(\frac{1}{2} + iT + iu)|^{4}T^{-s} \, dT \, e^{-u^{2}} \, du \right|^{2} dt \]

\[ + 1. \]

Both mean square integrals above are estimated analogously. The first one is, by using (2.8),

\[ \ll_{\varepsilon} 1 + \int_{V}^{2V} \int_{1}^{X} |\zeta(\frac{1}{2} + ix)|^{4}x^{-\sigma} \int_{x^{-x^{-T^{\varepsilon-1}}}^{x^{-x^{-T^{\varepsilon-1}}}} |\zeta(\frac{1}{2} + iy)|^{4}y^{-\sigma} \, dy \, dx \]

\[ \ll_{\varepsilon} 1 + \int_{V}^{2V} \int_{1}^{X} |\zeta(\frac{1}{2} + ix)|^{4}x^{-2\sigma}(xV^{\varepsilon-1} + x^{\varepsilon+c}) \, dx \, dt \]

\[ \ll_{\varepsilon} V^{c}(X^{2-2\sigma} + V + VX^{1+c-2\sigma}) \ll_{\varepsilon} V^{c}(X + VX^{c}). \quad (2.9) \]
Here we used (1.1), (2.2) the weak form of the fourth moment of $|\zeta(1/2 + ix)|$ and the bound (see (1.3))

$$E_2(T) \ll_{\epsilon} T^{c+\epsilon} \quad (1/2 \leq c \leq 3/4). \tag{2.10}$$

To estimate the contribution of $I_2(s, X)$, note that from [9, (4.10)] we have that the relevant part of $I_2(s, X)$ is, on integrating by parts,

$$
\int_0^b \int_0^\infty E_2'(\tau) f(\tau, \alpha) \, d\tau \, d\alpha = O \left( \sup_\alpha |E_2(X,f(X,\alpha)| \right) - \int_0^b \int_X^\infty E_2(\tau) \frac{\partial f(\tau, \alpha)}{\partial \tau} \, d\tau \, d\alpha.
$$

where $b > 0$ is a small constant, and $f(\tau, \alpha)$ is precisely defined in [9]. It was shown there that, for $0 < \sigma < 1/2, t \ll V$, we have the estimates

$$f(\tau, \alpha) \ll \tau^{2\xi-2-\sigma}(\log^2 \tau + V \log \tau + V^2) \log^3 \tau
$$

and

$$\frac{\partial f(\tau, \alpha)}{\partial \tau} \ll \tau^{2\xi-3-\sigma} V \log^3 \tau (\log^2 \tau + V \log \tau + V^2).$$

We use (2.5), (2.9), (2.10) and the above estimates to obtain, if $\sigma$ satisfies (2.2),

$$
\int_V^{2V} |I_1(s, X)|^2 \, dt \ll_{\epsilon} V^5 X^{2\xi+c-4-2\sigma} + V^6 \int_X^\infty E_2'(\tau) \tau^{\xi-5-2\sigma} \, d\tau \\
\ll_{\epsilon} V^\epsilon (V^5 X^{2\xi+c-4} + V^6 X^{4\xi-4}).
$$

It follows that

$$
\int_V^{2V} |Z_{22}(\sigma + it)|^2 \, dt \ll_{\epsilon} V^\epsilon (V X^c + X + V^5 X^{2\xi+c-5} + V^6 X^{4\xi-4}) \\
\ll_{\epsilon} V^\epsilon (V^{5/(4+c-\xi)} + V^{(4+c-\xi)/(4+\xi-\epsilon)} + V^{(15c-5)/(4+c-\epsilon)}) \\
\ll_{\epsilon} V^\epsilon (4+c-\xi)/(4+c-\epsilon) + c
$$

with $X = V^{5/(4+c-\xi)}$, since in view of $\xi \leq 1/2, 1/2 \leq c \leq 2/3$ we have

$$
5 \leq 4 + 6c - 4\xi. \quad 15c - 5 \leq 4 + 6c - 4\xi.
$$

Then with $\xi = 1/3$, which we henceforth assume, we obtain

$$
\int_V^{2V} |Z_{22}(\sigma + iv)|^2 \, dv \ll_{\epsilon} V^{2+\epsilon},
$$
Integral of the fourth moment of the Riemann zeta-function

\[ \int_{V}^{2V} |Z_{2t}(\nu + i\nu)|^2 d\nu \ll \varepsilon V^{3+\varepsilon}. \] (2.12)

It was shown in [9] that the major contribution to \( Z_{2\tau}(s) \) comes from \( (s = \sigma + it, V \leq t \leq 2V \) and \( \sigma \) satisfies (2.2))

\[ \sum_{t - V \leq \kappa_j \leq t + V} \alpha_j H_j^{3}(\tfrac{1}{2})|\frac{1}{2} + i\kappa_j - s|^{-1} \kappa_j^{-\frac{1}{2}} \int_{T(\kappa_j)}^{\infty} M^*(\kappa_j; T) T^{\frac{1}{2} + i\kappa_j} dT, \] (2.13)

where

\[ T(r) := r^{\frac{1}{1 + \varepsilon}} \log^{-D} r = r^{\frac{1}{2}} \log^{-D} r \quad (D > 0). \] (2.14)

and \( M^*(r; T) \) is a precisely defined function from spectral theory which satisfies, for \( T \geq T(r) \) (cf. [9, (4.28)]), the bound

\[ M^*(r; T) \ll r T^{-2} + r^{2+\varepsilon} T^{2\varepsilon - 3}. \] (2.15)

Thus the major contribution to the integral in (2.13) will therefore be, since \( H_j^{3}(\frac{1}{2}) \geq 0 \) (see Katz–Sarnak [10]),

\[ \int_{V}^{2V} \left| \sum_{t - V \leq \kappa_j \leq t + V} \alpha_j H_j^{3}(\tfrac{1}{2})|\frac{1}{2} + i\kappa_j - s|^{-1} V^{-\frac{1}{2}} \times \right. \]

\[ \left. \times \int_{T(V)}^{\infty} M^*(\kappa_j; T) T^{\frac{1}{2} + i\kappa_j} dT \right|^2 dt. \] (2.16)

Recall that \( \sigma \) is given by (2.2), and that by the zero-free region for \( \zeta(s) \) we have the bound (see [2, Lemma 12.3] and (2.2))

\[ \frac{1}{\zeta(\alpha + it)} \ll (\log t)^{2/3} (\log \log t)^{1/3} \quad (\alpha \geq 1 - \delta(t), t \geq t_0 > 0). \]

This gives \( |\frac{1}{2} + i\kappa_j - s|^{-1} \ll \log V \) in (2.16). We use the Cauchy-Schwarz inequality, (1.8) and the asymptotic formula (see [13])

\[ \sum_{\kappa_j \leq K} \alpha_j H_j^{3}(\tfrac{1}{2}) = (A \log K + B)K^2 + O(K \log^6 K) \quad (A > 0) \]

to estimate sums of \( \alpha_j H_j^{3}(\frac{1}{2}) \) in short intervals. We obtain then that the expression in (2.16) is, on using (2.9) and the inequality \( |g(x)g(y)| \leq \frac{1}{2}(g^2(x) + g^2(y)) \), (2.14)
and (2.15).

\[
\ll V^{-1} \log^2 V \int_V^{2V} \sum_{t-V^v \leq \kappa_j \leq t+V^v} \alpha_j H_j^2 (\frac{1}{2}) \sum_{t-V^v \leq \kappa_j \leq t+V^v} \alpha_j H_j^2 (\frac{1}{2}) \times \\
\int_{T(V)}^{\infty} M^*(\kappa_j; T) T^{\frac{1}{4} + i\kappa_j - \varepsilon} dT \bigg|_{T(V)}^2 dt
\]

\[
\ll V^\varepsilon \sum_{V-V^v \leq \kappa_j \leq 2V+V^v} \alpha_j H_j^4 (\frac{1}{2}) \int_{T(V)}^{\infty} |M^*(\kappa_j; T)|^2 T^{2-2\varepsilon} dT
\]

\[
\ll V^\varepsilon \sum_{V-V^v \leq \kappa_j \leq 2V+V^v} \alpha_j H_j^4 (\frac{1}{2}) \int_{T(V)}^{\infty} (V^2 T^{-4} + V^4 T^{4\kappa-6}) dT
\]

\[
\ll V^\varepsilon \sum_{V-V^v \leq \kappa_j \leq 2V+V^v} \alpha_j H_j^4 (\frac{1}{2}) \left( V^2 T^{-2}(V) + V^4 T^{4\kappa-4}(V) \right)
\]

\[
\ll V^\varepsilon \sum_{V-V^v \leq \kappa_j \leq 2V+V^v} \alpha_j H_j^4 (\frac{1}{2}) \ll V^{2+\varepsilon}.
\]

This establishes (2.12) and thus finishes the proof of Theorem 2.1. \(\square\)

3. The proof of Theorem 1.1

In this section we shall prove Theorem 1.1. The starting point is the inversion formula

\[
|\zeta(\frac{1}{2} + ix)|^4 = \frac{1}{2\pi i} \int_{(1)} \mathcal{Z}_2(s)x^{s-1} ds, \quad (3.1)
\]

where as usual \(\int_{(c)} = \lim_{T \to \infty} \int_{c-iT}^{c+iT}\). Namely, if \(F(s) = \int_0^\infty f(x)x^{-s} dx\) is the Mellin transform of \(f(x)\), \(y^r f(y) \in L^1(0, \infty)\) and \(f(y)\) is of bounded variation in a neighbourhood of \(y = x\), then one has the Mellin inversion formula (see [14])

\[
\frac{f(x) + f(x) - f(x-0)}{2} = \frac{1}{2\pi i} \int_{(c)} F(s)x^{-s} ds.
\]

We use this formula with \(f(x) = \frac{1}{x}|\zeta(\frac{1}{2} + \frac{i}{x})|^4\) for \(0 < x \leq 1\) and \(f(x) = 0\) for \(x > 1\), and then change \(x\) to \(1/x\) to obtain (3.1).

Now we replace the line of integration in (3.1) by the contour \(\mathcal{C}\), consisting of the same straight line from which the segment \([1 + \varepsilon - i, 1 + \varepsilon + i]\) is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole \(s = 1\) of the integrand. By the residue theorem we have

\[
|\zeta(\frac{1}{2} + i x)|^4 = \frac{1}{2\pi i} \int_{\mathcal{C}} \mathcal{Z}_2(s)x^{s-1} ds + Q_4(\log x) \quad (x > 1), \quad (3.2)
\]
where we have, since the coefficients of \( P_4(z) \) are naturally connected to the principal part of the Laurent expansion of \( Z_2(s) \) at \( s = 1 \) (see [3] and [13]),

\[
Q_4(\log x) = P_4(\log x) + P'_4(\log x)
\]

and \( P_4(x) \) is given by (1.1) and (1.2). If we integrate (3.2) from \( x = 1 \) to \( x = T \) and take into account the defining relation (1.1) of \( E_2(T) \), we shall obtain

\[
E_2(T) = \frac{1}{2\pi i} \int_{C} Z_2(s) \frac{T^s}{s} \, ds + O(1) \quad (T > 1).
\]  

(3.3)

A further integration, coupled with the deformation of the contour, enables one to deduce from (3.3) the formula

\[
\int_0^T E_2(t) \, dt = \frac{1}{2\pi i} \int_{(c)} Z_2(s) \frac{T^{s+1}}{s(s+1)} \, ds + O(T) \quad \left( \frac{1}{2} < c < 1, \, T > 1 \right),
\]  

(3.4)

since in view of the bound (see [9])

\[
\int_0^T |Z_2(\sigma + it)|^2 \, dt \ll T^{2+\epsilon} \quad \left( \frac{1}{2} < \sigma < 1 \right)
\]  

(3.5)

we may take \( \frac{1}{2} < c < 1 \) as the range for \( c \) in (3.4). The formula (3.4) is the key one in the proof of Theorem 1.1. We replace the line of integration in the integral on the right-hand side of (3.4) by the contour consisting of the segment \([\sigma_0 - it_0, \sigma_0 + it_0] \), and the curve

\[
\sigma = \frac{1}{2} - C\delta(|t|), \quad \delta(x) := (\log x)^{-2/3}(\log \log x)^{-1/3}, \quad |t| \geq t_0, \quad \sigma_0 = \frac{1}{2} - C\delta(t_0),
\]  

(3.6)

where \( C \) denotes positive possibly different constants. Since \( Z_2(x) \) has poles at complex zeros of \( \zeta(2s) \) it follows, by the strongest known zero-free region for \( \zeta(s) \) (see [6, Chapter 6]), that the function \( Z_2(s) \) is regular on the new contour. The residue theorem yields

\[
\int_0^T E_2(t) \, dt = 2\Re \left\{ \sum_{j=1}^{\infty} \frac{T^{\frac{3}{2} + i\kappa_j}}{(\frac{1}{2} + i\kappa_j)((\frac{1}{2} + i\kappa_j) - \kappa_j)^3} H_{\frac{1}{2}}(1) R_1(\kappa_j) \right\} + O(T^{\sigma_0 + 1}) \]

\[
+ O\left( \int_{t_0}^{\infty} T^{\frac{3}{2} - C\delta(t) - 2}\left| Z_2(\frac{1}{2} - C\delta(t) + it) \right| \, dt \right)
\]  

(3.7)

with \( R_1(\kappa_j) \) given by (1.12). Let \( \eta(T) \) be defined by (1.11) and put

\[
U = U(T) := e^{C\eta(T)} = e^{C \log^{3/2} T \log \log T}.
\]
Then
\[
\int_{t_0}^\infty - \int_U^{t_0} + \int_U^\infty \ll T^{3/2} e^{-C\delta(U) \log T} + T^{3/2} U^{-\frac{1}{4}} + T^{3/2} e^{-C\gamma(T)}.
\] (3.8)
since by Theorem 2.1 we have
\[
\int_V^{2V} |Z_2(\frac{1}{2} + C\delta(v) + iv)|^2 dv \ll V^{2+\varepsilon}.
\] (3.9)
Namely we split the integral in the O-term in (3.7) into subintegrals over \([V, 2V]\). The contour \(\sigma = \frac{1}{2} - C\delta(v)\) is replaced by \(\sigma = \frac{1}{2} - C\delta(V)\), which is technically easier. In this process we obtain integrals over horizontal segments whose contributions will be \(\ll_{v} V^{2+\varepsilon}\), since by (5.10) and (5.24) of [9] (with \(\xi = \frac{1}{3}\)) we have the bound
\[
Z_2(\frac{1}{2} - C\delta(v) + iv) \ll_{v} v^{1+\varepsilon}.
\]
Finally by the Cauchy-Schwarz inequality for integrals and (3.9) we obtain
\[
\int_{V}^{\infty} |Z_2(\frac{1}{2} - C\delta(v) + iv)|v^{-2} dv \ll 1.
\]
\[
\int_{V}^{\infty} |Z_2(\frac{1}{2} - C\delta(v) + iv)|v^{-2} dv \ll_{v} V^{\varepsilon-\frac{1}{2}}.
\]
thereby establishing (3.8) and completing the proof of Theorem 1.1.

In concluding it may be remarked that, similarly as in [5], one may obtain quickly from (3.3) the bound (see (1.3))
\[
E_2(T) \ll_{\varepsilon} T^{\frac{5}{2}+\varepsilon}.
\] (3.10)
which is (up to \(\varepsilon\)) the strongest one known. Namely by [5, (5.3)] we have
\[
E_2(T) \leq C_1 H^{-1} \int_T^{T+H} E_2(x) f(x) dx + C_2 H \log^4 T
\] (3.11)
where \(f(x) (> 0)\) is a smooth function supported in \([T, T + H]\), such that \(f(x) = 1\) for \(T + \frac{1}{2} H \leq x \leq T + \frac{3}{2} H\). Then from (3.3) we have \((\frac{1}{2} < c < 1)\)
\[
E_2(T) \leq \frac{C_1}{2\pi i H} \int_{(c)} Z_2(s) \int_T^{T+H} f(x)x^{-s} dx ds + C_2 H \log^4 T.
\]
We take \(c = \frac{3}{4} + \varepsilon\), use (3.5), the Cauchy-Schwarz inequality, and the fact that by \(r\) integrations by parts it follows that
\[
\int_T^{T+H} f(x)x^s dx = (-1)^r \int_T^{T+H} \frac{x^{s+r}}{(s+1)\cdots(s+r)} f^{(r)}(x) dx \ll_{\varepsilon, r} T^{\sigma+r} H^{1-r} |H|^{-r}.
\]
Hence the above integral over \( s \) may be truncated at \( |\text{Im} \ s| = T^{1+\varepsilon} H^{-1} \) with a negligible error, and we obtain

\[
E_2(T) \ll_{\varepsilon} T^{\frac{4}{3}+\varepsilon} \int_1^{T^{1+\varepsilon} H^{-1}} |Z_2(\frac{1}{2} + \varepsilon + it)| \frac{dt}{t} + H \log^4 T
\]

\[
\ll_{\varepsilon} T^{\varepsilon} (TH^{-\frac{1}{2}} + H) \ll T^{\frac{4}{3}+\varepsilon}
\]

with \( H = T^{2/3} \). A lower bound for \( E_2(T) \), similar to (3.11), also holds, and therefore (3.10) follows as asserted.

References


**Address:** Aleksandar Ivić, Katedra Matematike RGF-a, Universiteta u Beogradu, Djušina 7, 11000 Beograd, Serbia (Yugoslavia)

**E-mail:** aleks@ivic.matf.bg.ac.yu, aivic@rgf.rgf.bg.ac.yu

**Received:** 14 September 1999