ON SUMS OF THREE UNIT FRACTIONS WITH POLYNOMIAL DENOMINATORS
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Abstract: The equation \( \frac{m}{(ax + b)} = \frac{1}{F_1(x)} + \frac{1}{F_2(x)} + \frac{1}{F_3(x)} \) is shown to be impossible under some conditions on polynomials \( ax + b \) and \( F_1, F_2, F_3 \).

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A well known conjecture of Erdős and Straus [2] asserts that for every integer \( n > 1 \) the equation

\[
\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}
\]

is solvable in positive integers \( x_1, x_2, x_3 \). Sierpiński [10] has made an analogous conjecture concerning \( \frac{5}{n} \) and the writer has conjectured that for every positive integer \( m \) the equation

\[
\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}
\]

is solvable in positive integers \( x_1, x_2, x_3 \) for all integers \( n > n_0(m) \) (see [10], p. 25). For \( m \leq 12 \) one knows many identities

\[
\frac{m}{ax + b} = \frac{1}{F_1(x)} + \frac{1}{F_2(x)} + \frac{1}{F_3(x)}
\]

where \( a, b \) are integers, \( a > 0 \) and \( F_i \) are polynomials with integral coefficients and the leading coefficients positive, see [1], [5], [7], [8], [11], Section 28.5. It could seem that a proof of solvability of (2) for a fixed \( m \) and \( n > n_0(m) \) could be obtained by producing a finite set of identities of the form (2) with a fixed \( a \) and \( b \) running through the set of all residues mod \( a \). The theorems given below show that this is impossible.

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Theorem 1. Let \( a, b \) be integers, \( a > 0 \), \( (a, b) = 1 \). If \( b \) is a quadratic residue \( \mod a \), then there are no polynomials \( F_1, F_2, F_3 \) in \( \mathbb{Z}[x] \) with the leading coefficients positive, satisfying (2) with \( m \equiv 0 \mod 4 \).

Theorem 2. Let \( m, a, b \) be integers, \( a > 0 \), \( m > 3b > 0 \). There are no polynomials \( F_1, F_2, F_3 \) in \( \mathbb{Z}[x] \) with the leading coefficients positive, satisfying (2).

Theorem 1 in the crucial case \( m = 4 \) has been quoted in the book [4] (earlier inaccurately in [3]), but the proof has not been published before. The theorem is closely related to a result of Yamamoto [12] and the crucial lemma is a consequence of his work. Possibly, Theorem 2 can be generalized as follows. Let \( k, m, a, b \) be positive integers, \( m > kb \). There are no polynomials \( F_1, F_2, \ldots, F_k \) in \( \mathbb{Z}[x] \) with the leading coefficients positive such that

\[
\frac{m}{ax + b} = \sum_{i=1}^{k} \frac{1}{F_i(x)}.
\]

Note that by a theorem of Sander [9] the above equation has only finitely many solutions in polynomials \( F_i \) for fixed \( a, b, m \) and \( k \).

**Notation.** For \( \Omega \subset \mathbb{R}[x] \) we shall denote by \( \Omega^+ \) the set of polynomials in \( \Omega \) with the leading coefficient positive.

For two polynomials \( A, B \) in \( \mathbb{Z}[x] \), not both zero, we shall denote by \( (A, B) \) the polynomial \( D \in \mathbb{Z}[x]^+ \) with the greatest possible degree and the greatest possible leading coefficient such that \( A/D \in \mathbb{Z}[x] \) and \( B/D \in \mathbb{Z}[x] \).

**Lemma 1.** If \( A, B, C, D \) are in \( \mathbb{Z}[x] \), \( (A, B) = 1 \) and \( A/B = C/D \), then \( C = HA, \ D = HB \) for an \( H \in \mathbb{Z}[x] \). If \( (C, D) = 1 \) then \( H = \pm 1 \).

**Proof.** This follows from Theorem 44 in [6], the so called Gauss’s lemma. \( \blacksquare \)

**Lemma 2.** The equations

\[
n^2 = 4(cs - b^*)b' r - s \tag{3}
\]

and

\[
n^2 s = 4(cs - b^*)b'^* r - 1 \tag{4}
\]

have no solutions in positive integers \( b^*, c, n, r, s \).

**Proof.** This is a consequence of Theorem 2 in [12]: according to this theorem \( n^2 \) does not satisfy either of the two congruences

\[
n^2 \equiv -s(\mod 4a^*b^*), \tag{5}
\]

\[
n^2 s \equiv -1(\mod 4a^*b^*), \tag{6}
\]

where \( a^*, b^*, s \) are positive integers and \( s \mid a^* + b^* \), while just such congruences follow from (3) and (4) with \( a^* = cs - b^* \). The impossibility of the congruences
(5) and (6) is established in [12] by evaluation of the Kronecker symbol \((-s/ab)\); instead one can use the Jacobi symbol as follows.

(3) gives \(n^2 = (4b^*cr - 1)s - 4b^*s^2r\), (4) gives \((ns)^2 = (4b^*crs - 1)s - 4b^*rs^2\), while for \(e = 2^\alpha e_0 > 0, e_0\) odd, we have by the reciprocity law ([6], Section 42)

\[
\left( \frac{-4b^*e}{4b^*es - 1} \right) = -\left( \frac{e_0}{4b^*es - 1} \right) = -(-1)^{(e_0-1)/2} \left( \frac{4b^*es - 1}{e_0} \right) \\
= -(-1)^{(e_0-1)/2} \left( \frac{-1}{e_0} \right) = -1.
\]

\[\blacksquare\]

**Proof of Theorem 1.** It is clearly sufficient to prove the theorem for \(m = 4\). Assume that we have (2) with \(m = 4\). Thus

\[4F_1(x)F_2(x)F_3(x) = (ax + b)(F_2(x)F_3(x) + F_1(x)F_4(x) + F_1(x)F_2(x))\]

hence

\[F_1(\ b/a)F_2(\ b/a)F_3(\ b/a) = 0.\]

If we had \(F_i(-b/a) = 0\) for each \(i \leq 3\), then there would exist polynomials \(G_i \in \mathbb{Q}[x]^+\) such that \(F_i(x) = (ax + b)G_i(x)\). Since \((a, b) - 1\) it follows from Gauss's lemma that \(G_i \in \mathbb{Z}[x]^+\). Choosing an integer \(k\) such that \((ak + b)G_1(k)G_2(k)G_3(k) \neq 0\) we should obtain

\[4 = \frac{1}{G_1(k)} + \frac{1}{G_2(k)} + \frac{1}{G_3(k)} \leq 3, \quad \text{a contradiction.}\]

Hence, up to a permutation of \(F_1, F_2, F_3\) there are two possibilities

\[F_1(-b/a) = F_2(-b/a) = 0 \neq F_3(-b/a), \quad (7)\]

\[F_1(-b/a) = 0 \neq F_2(-b/a)F_3(-b/a) \quad (8)\]

In the case (7) \(F_i(x) = (ax + b)G_i(x)\) \((i = 1, 2)\), \((F_3(x), ax + b) = 1\), where \(G_i \in \mathbb{Z}[x]^+\). Let us put

\[D = (G_1, G_2), \quad G_i = DH_i \quad (i = 1, 2),\]

\[C = (4DH_1H_2 - H_1 - H_2, DH_1H_2) = (H_1 + H_2, D),\]

\[D = CR, \quad H_1 + H_2 = CS.\]

\(H_i, C, R, S\) are in \(\mathbb{Z}[x]^+\) and we have \((H_1, H_2) = 1\), \((RH_1H_2, S) = 1\). By (2) with \(m = 4\)

\[ax + b \quad F_3 \quad \frac{4DH_1H_2 - H_1 - H_2}{DH_1H_2} = \frac{4RH_1H_2 - S}{RH_1H_2}.\]
Since \((ax + b, F_3) = 1 = (4RH_1H_2 - S, RH_1H_2)\) and both \(F_3\) and \(RH_1H_2\) are in \(\mathbb{Z}[x]^+\), it follows by Lemma 1 that
\[
ax + b = 4RH_1H_2 - S = 4(CS - H_2)H_2R - S. \tag{9}
\]

Since \(b\) is a quadratic residue for \(a\) and \(C, H_2, R, S\) are in \(\mathbb{Z}[x]^+\) there exist integers \(k\) and \(n\) such that
\[
ax + b = n^2 \quad \text{and} \quad b^* = H_2(k), \ c = C(k), \ r = R(k), \ s = S(k) \quad \text{are in } \mathbb{Z}^+,
\]
which in view of (9) contradicts Lemma 2.

Consider now the case (8). We have here
\[
F_i(x) = (ax + b)G_1(x), \ F_i = DH_i \ (i = 2, 3)
\]
where \(G_1 \in \mathbb{Z}[x]^+, \ D = (F_2, F_3), \ (H_2, H_3) = 1\) and \((DH_i, ax + b) = 1 \ (i = 2, 3), \ H_i \in \mathbb{Z}[x]^+\). Hence, by (2) with \(m = 4\)
\[
\frac{4}{ax + b} = \frac{1}{(ax + b)G_1} + \frac{H_2 + H_3}{DH_2H_3},
\]
\[
DH_2H_3 = \frac{G_1(H_2 + H_3)}{4G_1 - 1}. \tag{10}
\]

Let us put \(C = (D, H_2 + H_3), \ D = CR, \ H_2 + H_3 = CS\), so that \(C, R, S\) are in \(\mathbb{Z}[x]^+\). Since \((DH_2H_3, ax + b) = 1\) we infer from Lemma 1 that \(4G_1 - 1 = (ax + b)H_1\), where \(H_1 \in \mathbb{Z}[x]^+\). Hence, by (10),
\[
\frac{RH_2H_3}{S} = \frac{G_1}{H_1}.
\]

Since \((RH_2H_3, S) = 1 = (G_1, H_1)\) and \(S\) and \(H_1\) are in \(\mathbb{Z}[x]^+\) it follows from Lemma 1 that \(H_1 = S, \ G_1 = RH_2H_3\) and
\[
(ax + b)S = 4G_1 - 1 = 4RH_2H_3 - 1 = 4(CS - H_2)H_2R - 1. \tag{11}
\]

Since \(b\) is a quadratic residue \(mod\ a\) and \(C, H_2, R, S\) are in \(\mathbb{Z}[x]^+\) there exist integers \(k\) and \(n\) such that
\[
ax + b = n^2 \quad \text{and} \quad b^* = H_2(k), \ c = C(k), \ r = R(k), \ s = S(k) \quad \text{are in } \mathbb{Z}^+.
\]
which in view of (11) contradicts Lemma 2.

**Proof of Theorem 2.** If \(F_i(0) \neq 0\) for all \(i\) it follows from (2) on substituting \(x = 0\) that
\[
\frac{m}{b} = \sum_{i=1}^{3} \frac{1}{F_i(0)} \leq 3,
\]
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contrary to the assumption $m > 3b$.

If $F_i(0) \neq 0$ for all but one $i$, it follows from (2) on taking the limit for $x \to 0$

$$\frac{m}{b} = \pm \infty,$$

a contradiction.

If $F_i(0) = 0$ for all $i$, it follows $F_i(x) = xG_i(x)$, $G_i \in \mathbb{Z}[x]^+$ and by (2)

$$\frac{mx}{ax + b} = \sum_{i=1}^{3} \frac{1}{G_i(x)}.$$

When $x \to \infty$ the terms on the left hand side are less than the limit $m/a$, the terms on the right hand side are greater or equal to the limit, which contradicts the equality.

Thus $F_i(0) = 0$ for exactly two $i \leq 3$ and we may assume without loss of generality that

$$F_i(0) = 0 \quad (i = 1, 2), \quad F_3(0) \neq 0.$$

Arguing as in the proof of Theorem 1 we infer that $F_i(-b/a) = 0$ for at least one $i$. Hence up to a permutation of $F_1, F_2$ there are the following possibilities:

(12) \hspace{1cm} F_i(-b/a) = 0 \quad (i = 1, 2, 3);

(13) \hspace{1cm} F_i(-b/a) = 0 \quad (i = 1, 2), \quad F_3(-b/a) \neq 0;

(14) \hspace{1cm} F_i(-b/a) = 0 \quad (i = 1, 3), \quad F_2(-b/a) \neq 0;

(15) \hspace{1cm} F_i(-b/a) \neq 0 \quad (i = 1, 2), \quad F_3(-b/a) = 0;

(16) \hspace{1cm} F_i(-b/a) \neq 0 \quad (i = 1, 3), \quad F_2(-b/a) = 0.

We shall consider these cases successively.

**Case (12).** Here $F_i(x) = (ax + b)G_i(x)$, $G_i \in \mathbb{Q}[x]^+$ \quad ($i = 1, 2, 3$) and by Gauss's lemma $\langle a, b \rangle C_i \subset \mathbb{Z}[x]^+$. Taking an integer $k$ such that $G_i(k) \neq 0$ we obtain from (2)

$$m - \sum_{i=1}^{3} \frac{1}{G_i(k)} \leq 3(a, b) \leq 3b,$$

contrary to the assumption.

**Case (13).** Here $F_i(x) = x(ax + b)G_i(x)$, $G_i \in \mathbb{Q}[x]^+$ \quad ($i = 1, 2$)

$$m = \frac{1}{xG_1(x)} + \frac{1}{xG_2(x)} + \frac{ax + b}{F_3}$$

and taking the limit for $x \to \infty$ we infer that $F_3 = cx + d$, where $c = a/m$. Hence

$$0 = \frac{1}{xG_1} + \frac{1}{xG_2} + \frac{b - md}{ex + d}.$$
For $x$ large enough the first two terms are positive, hence $b - md < 0$ and $d > 0$.

Without loss of generality $G_2(-d/c) = 0$, hence $G_2 = (cx + d)H_2(x)$, $H_2 \in \mathbb{Q}[x]^+$,

$$0 = \lim_{x \to \infty} \frac{cx + d}{xG_1(x)} + b - md,$$

thus $G_1(x) = c/(md - b)$ and

$$0 = \frac{md - b}{cx} \frac{1}{x} + \frac{b - md}{x(cx + d)H_2} + \frac{(md - b)d}{x(cx + d)} + \frac{1}{x(cx + d)H_2}.$$

This is impossible, since for $x$ large enough both terms on the right hand side are positive.

**Case (14).** Here $F_1 = x(ax + b)G_1$, $F_2 = xG_2$, $F_3 = (ax + b)G_3$, where $G_i \in \mathbb{Q}[x]^+$ ($i = 1, 2, 3$) and

$$m = \frac{1}{G_1} + \frac{ax + b}{G_2} + \frac{1}{G_3}.$$

The first and the second term on the right hand side are greater than their limits for $x \to \infty$, the third term is greater or equal, while the left hand side is constant: this gives a contradiction.

**Case (15).** Here $F_1 = xG_i$, ($i = 1, 2$), $F_3 = (ax + b)G_3$, where $G_i \in \mathbb{Z}[x]^+$, $G_i(-b/a) \neq 0$ ($i = 1, 2$), $G_3 \in \mathbb{Q}[x]^+$ and

$$m = \frac{1}{G_1(x)} + \frac{1}{G_2(x)} + \frac{x}{(ax + b)G_3(x)}.$$

If $G_3 \not\in \mathbb{Q}^+$ all three terms on the right hand side are greater than or equal to their limits for $x \to \infty$, while the left hand side is less than the limit, a contradiction. Hence $G_3 = g \in \mathbb{Q}^+$ and

$$\frac{(m - 1/g)x}{ax + b} = \frac{1}{G_1} + \frac{1}{G_2},$$

which contradicts $G_1G_2(-b/a) \neq 0$.

**Case (16).** Here $F_1 = xG_1$, $F_2 = x(ax + h)G_2$, where $G_1 \in \mathbb{Z}[x]^+$, $G_2 \in \mathbb{Q}[x]^+$ and

$$m = \frac{1}{G_1} + \frac{1}{G_3} = \frac{x}{(ax + b)G_2} + \frac{x}{F_3}.$$  \hfill (17)

If $\deg F_3 = 0$ we take the limit for $x \to \infty$ and obtain $m/a = \infty$, a contradiction.

If $\deg F_3 > 1$, when $x \to \infty$ the left hand side of (17) is less than its limit, while all three terms on the right hand side are greater than or equal to their limits, which gives a contradiction. Thus

$$\deg F_3 = 1, \; F_3 = cx + d, \; \text{where} \; c \in \mathbb{Z}^+, \; d/c \neq b/a.$$  \hfill (18)

We consider four subcases:

(i) \quad $\deg G_1 > 1$;

(ii) \quad $\deg G_1 = 1, \; G_1/F_3 \not\in \mathbb{Q}$;

(iii) \quad $\deg G_1 = 1, \; G_1/F_3 \in \mathbb{Q}$;

(iv) \quad $\deg G_1 - 1 = 0$.
**Subcase (i).** Taking the limit for \( x \to \infty \) we infer from (17) and (18) that \( a = cm \)

\[
\frac{mx}{cmx + b} = \frac{1}{G_1} + \frac{1}{(cmx + b)G_2} + \frac{x}{cx + d},
\]

\[
\frac{x(md - b)}{cx + d} = \frac{cmx + b}{G_1} + \frac{1}{G_2},
\]

hence \( md - b > 0, \ d > 0 \). When \( x \to \infty \) the left hand side of (18) is less than its limit, while both terms on the right hand side are greater than or equal to their limits, which gives a contradiction.

**Subcase (ii).** As in the subcase (i) we have \( md - b > 0, \ d > 0 \). Let \( G_1 = ex + f \), \( e > 0 \), \( f/e \neq b/a, d/c \). It follows from (19) that

\[
G_2 = g^{-1}(cx + d)(ex + f), \ g \in \mathbb{Q}^+.
\]

and substituting \( x = 0 \) we obtain

\[
0 = \frac{b}{f} + \frac{g}{df}; \quad g = -bd < 0,
\]

a contradiction.

**Subcase (iii).** Let \( G_1 = e^{-1}(cx + d), \ e \in \mathbb{Q}^+ \). We obtain from (17) and (18)

\[
\frac{mx}{ax + b} = \frac{1}{(ax + b)G_2} + \frac{x + e}{cx + d},
\]

hence \( G_2 = f^{-1}(cx + d), \ f \in \mathbb{Q}^+ \) and substituting \( x = 0 \)

\[
0 = \frac{f}{bd} + \frac{e}{d}; \quad f = -be < 0,
\]

a contradiction.

**Subcase (iv).** Let \( G_1 = g \). It follows from (17) and (18) that \( G_2 = e^{-1}(cx + d), \ e \in \mathbb{Q}^+ \),

\[
\frac{mx}{ax + b} = \frac{1}{g} + \frac{e}{(ax + b)(cx + d)} + \frac{x}{cx + d}
\]

and multiplying both sides by \((ax + b)(cx + d)\)

\[
(cgm - ac - ag)x^2 + (dgm - bg - ad - bc)x - bd - e = 0.
\]

Hence

(20) \( cgm - ac - ag = 0 \),

(21) \( dgm - bg - ad - bc = 0 \),

(22) \( bd + e = 0 \),
which is impossible, since (20) gives \( gm - a = ag/c > 0 \), (21) gives \( d = (bg + bc)/(gm - a) > 0 \), contrary to (22).

References


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