FUNCTIONES ET APPROXIMATIO
XXXI (2003), 109-117

FACTORORIZATION IN THE EXTENDED SELBERG CLASS
JERZY KACZOROWSKI & ALBERTO PERELLI

Abstract: We prove that every function in the extended Selberg class \( S^d \) can be factored into primitive functions. The proof is definitely more involved than in the case of the Selberg class \( S \).

Keywords: extended Selberg class, factorization, general \( L \)-functions.

1. Introduction

We denote by \( S \) the Selberg class of Dirichlet series with functional equation and Euler product. It is well known that \( S \) contains several classical \( L \)-functions, and it is expected that \( S \) essentially coincides with the class of automorphic \( L \)-functions. We refer to the survey paper [5] for definitions, notation and basic properties of \( S \) and related classes of Dirichlet series, such as the extended Selberg class \( S^d \) of Dirichlet series with functional equation, but not necessarily with Euler product. We recall that a function \( F(s) \) in \( S \) is primitive if \( F(s) = F_1(s)F_2(s) \) with \( F_1, F_2 \in S \) implies \( F_1 = 1 \) or \( F_2 = 1 \). It is well known that every function in \( S \) can be factored into primitive functions; see Conrey-Ghosh [2]. The proof is an immediate consequence, by a simple induction on the degree, of the following three facts:

i) the degree is additive, i.e., \( d_{FG} = d_F + d_G \) for \( F, G \in S \);

ii) there are no functions \( F \in S \) with degree \( 0 < d_F < 1 \);

iii) the only function of degree 0 in \( S \) is the constant 1.

The notion of primitive function is defined in the extended Selberg class \( S^d \) as well, and hence the problem of the factorization into primitive functions can also be raised in the framework of \( S^d \). In view of Lemma 1 below, in this case we consider only factorizations up to constants, since the non-zero constants are invertible in \( S^d \). Note that the first two of the above facts still hold in \( S^2 \), see [4], but \( S^2_0 \) is not any more reduced to the single function \( F(s) = 1 \) identically. We refer to Theorem 1 of [4] for the characterization of functions in \( S^2_0 \). As a consequence, the above simple induction on the degree is not enough to show

2001 Mathematics Subject Classification: 11M41
the existence of the factorization in $S^d$. However, the argument can be suitably modified to prove the following

**Theorem 1.** Every function in the extended Selberg class $S^d$ can be factored into primitive functions.

The basic tools in the proof of Theorem 1 are the notion of conductor of $F \in S^d$ and the characterization of the functions of degree 0 in $S^d$, see [4]. Indeed, we recall that the conductor $q_F$ of $F \in S^d$ is defined as

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{k} \lambda_j^{2\lambda_j},$$

see [6]. Note that $q_F$ is multiplicative, i.e., $q_{FG} = q_F q_G$ if $F, G \in S^d$, and $q_F = Q^2$ if $F \in S^d_0$. Moreover, if $F \in S^d_0$ then $q_F$ is a positive integer and $F(s)$ is a Dirichlet polynomial of the form

$$F(s) = \sum_{n \mid q_F} a(n)n^{-s}. \quad (1.1)$$

Further, $S^d_d = \emptyset$ for $0 < d < 1$, and $F(s)$ is constant if and only if $d_F = 0$ and $q_F = 1$; see Theorem 1 of [4] for the above results.

We call almost-primitive a function $F \in S^d$ such that if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in S^d$, then $d_{F_1} = 0$ or $d_{F_2} = 0$. We have

**Theorem 2.** If $F \in S^d$ is almost-primitive, then $F(s) = G(s)P(s)$ with $G, P \in S^d$, $d_G = 0$ and $P(s)$ primitive.

We remark that Theorem 1 is a simple consequence of Theorem 2 and of the above recalled results. In fact, an induction on the degree shows that every $F \in S^d$ can be written as

$$F(s) = F_1(s) \cdots F_k(s),$$

where each $F_j(s)$ is almost-primitive. Therefore, by Theorem 2 we have

$$F(s) = G(s)P_1(s) \cdots P_k(s)$$

with primitive $P_j(s)$ and $d_G = 0$. Since the functions in $S^d_0$ have integer conductor and those with conductor equal to 1 are constant, an induction on the conductor shows that $G(s)$ is a product of primitive functions, and Theorem 1 follows.

A well known conjecture states that $S$ has unique factorization into primitive functions. Moreover, it is well known that the Selberg orthonormality conjecture implies such a conjecture; see section 4 of [5]. Note that the analog of the Selberg orthonormality conjecture does not hold in $S^d$. Indeed, let $\chi_1(n)$ and $\chi_2(n)$ be two primitive Dirichlet characters with the same modulus and parity, and consider
\[ F(s) = L(s, \chi_1) + L(s, \chi_2) \text{ and } G(s) = L(s, \chi_1). \] Then \( F(s) \) and \( G(s) \) belong to \( \mathcal{S}^\sharp \) and are primitive (since the functions of \( S \) are linearly independent over the \( p \)-finite Dirichlet series, see [3]), but it is easily checked that the Selberg orthonormality conjecture does not hold for \( F(s) \) and \( G(s) \). It remains open the problem of determining if the unique factorization holds in \( \mathcal{S}^\sharp \). We conclude with another interesting problem related with the factorization in \( \mathcal{S}^\sharp \): is it true that a primitive function in \( S \) is primitive in \( \mathcal{S}^\sharp \) as well?

**Acknowledgments.** We wish to thank Alessandro Zaccagnini for suggesting some improvements in the presentation of the paper. This research was partially supported by the Istituto Nazionale di Alta Matematica, by a MURST grant and the KBN grant 2 PO3A 024 17.

2. Proof of Theorem 2

We first characterize the invertible elements of \( \mathcal{S}^\sharp \).

**Lemma 1.** The invertible functions in \( \mathcal{S}^\sharp \) are the non-zero constants.

**Proof.** Clearly, the non-zero constants are invertible in \( \mathcal{S}^\sharp \). Let now \( F \in \mathcal{S}^\sharp \) be invertible, and let \( G(s) = F(s)^{-1} \). Then \( d_F + d_G = 0 \), and hence both \( F(s) \) and \( G(s) \) are Dirichlet polynomials (see the Introduction). Denoting by \( n_0 \) and \( m_0 \) the largest indexes of non-zero coefficients of \( F(s) \) and \( G(s) \), respectively, we have that the coefficient of index \( n_0m_0 \) of \( F(s)G(s) \) is non-zero. Therefore \( n_0m_0 = 1 \), and Lemma 1 follows. \( \blacksquare \)

It is well known that every \( F \in \mathcal{S}^\sharp \) has a zero-free half-plane, say \( \sigma > \sigma_F \). By the functional equation, \( F(s) \) has no zeros for \( \sigma < -\sigma_F \), apart from the trivial zeros coming from the poles of the \( \Gamma \)-factors. We denote by \( \rho = \beta + i\gamma \) the generic zero of \( F(s) \), and write

\[ N_F(T) = \# \{ \rho : F(\rho) = 0, |\beta| \leq \sigma_F, |\gamma| < T \}. \]

The classical proof of the Riemann-von Mangoldt formula can be adapted to show that

\[ N_F(T) = \frac{d_F}{\pi}T \log T + c_F T + O(\log T) \quad (2.1) \]

with a certain constant \( c_F \), for \( T \geq 2 \) and any fixed \( F \in \mathcal{S}^\sharp \) with \( d_F > 0 \); see section 2 of [5]. The proof of Theorem 2 is based on the following uniform estimate for the number of zeros of functions in \( \mathcal{S}^\sharp_0 \), i.e., Dirichlet polynomials of type (1.1).

**Proposition 1.** We have

\[ N_F(T) = \frac{T}{\pi} \log q_F + O_{\sigma_F}(\log \log q_F) \]

uniformly for \( T \geq 2 \) and \( F \in \mathcal{S}^\sharp_0 \) with \( a(1) = 1 \) and \( q_F \geq 2 \).
A similar result already appears as Proposition 1 of Bombieri-Friedlander [1]. However, Proposition 1 of [1] deals with more general Dirichlet polynomials but gives only an upper bound for $N_F(T)$, while we need a lower bound. We first show how Theorem 2 follows from our Proposition 1, and in the next section we prove Proposition 1.

Assume that $F \in S^d$ is almost-primitive. If $F(s)$ is not primitive, it can be written as

$$F(s) = L_1(s)F_1(s)$$

with $d_{L_1} = 0$, $q_{L_1} \geq 2$ and $F_1(s)$ almost-primitive. If $F_1(s)$ is not primitive we apply inductively the same reasoning, and hence arguing by contradiction we may assume that for every $n \in \mathbb{N}$

$$F(s) = L_1(s) \cdots L_n(s)F_n(s)$$  \hspace{1cm} (2.2)

with $d_{L_j} = 0$, $q_{L_j} \geq 2$ and $F_n(s)$ almost-primitive, $j = 1, \ldots, n$. Moreover, looking at the Dirichlet series of both sides of (2.2), we have that only a finite number of $L_j(s)$ can have first coefficient $a_{L_j}(1) = 0$. Therefore, by a normalization, for $n$ sufficiently large we can rewrite (2.2) as

$$F(s) = H(s)H_1(s) \cdots H_n(s)F_n(s)$$

with $d_H = 0$, $d_{H_j} = 0$, $q_{H_j} \geq 2$, $a_{H_j}(1) = 1$ and $F_n(s)$ almost-primitive, $j = 1, \ldots, n$. Writing $G_n(s) = H_1(s) \cdots H_n(s)$, for large $n$ we finally obtain

$$F(s) = H(s)G_n(s)F_n(s)$$  \hspace{1cm} (2.3)

with $d_H = 0$, $d_{G_n} = 0$, $q_{G_n} \to \infty$ as $n \to \infty$, $a_{G_n}(1) = 1$ and $F_n(s)$ almost-primitive.

Since the conductor of the functions in $S^d_0$ is integer and $S^d_d = \emptyset$ for $0 < d < 1$, from (2.2) we immediately have that $d_F \geq 1$. Hence we may use (2.1) and Proposition 1 to show that (2.3) is impossible. Indeed, for $n$ sufficiently large we have

$$N_F(T) \geq N_{G_n}(T),$$

and $G_n(s) \neq 0$ for $\sigma > \sigma_F$. Therefore, from (2.1) and Proposition 1 we have

$$\frac{d_F}{\pi} T \log T \geq \frac{1}{2\pi} T \log q_{G_n} + O(\log^6 q_{G_n})$$

for sufficiently large $T$, and hence we get a contradiction as $n \to \infty$ by choosing $T = T_n = q_{G_n}^\delta$ with a small $\delta > 0$. 
3. Proof of Proposition 1

Since \( a(1) = 1 \), we can find a sufficiently large \( \sigma_0 > \sigma_F \) such that

\[
|F(s) - 1| \leq \frac{1}{4} \quad \text{for} \quad \sigma \geq \sigma_0;
\]

we will choose \( \sigma_0 \) later on. Moreover, we may assume that \( \pm T \) is not the ordinate of a zero of \( F(s) \) and that \( q_F \geq 2 \). Recalling that \( q_F = Q^2 \) for \( F \in S_0^2 \), by a standard technique based on the argument principle, the functional equation and (3.1) we have

\[
N_F(T) = \frac{1}{2\pi} \Delta_R \arg (Q^*F(s)) = \frac{1}{\pi} \Delta_{L_1 \cup L_2 \cup L_3} \arg (Q^*F(s)) \\
= \frac{T}{\pi} \log q_F + O(1) + O(|\Delta_{L_1 \cup L_3} \arg (Q^*F(s))|),
\]

where \( R \) is the rectangle of vertices \( \sigma_0 \pm iT, \ 1 - \sigma_0 \pm iT \) and \( L_1 \cup L_2 \cup L_3 \) is the right half of its perimeter, \( L_2 \) being the vertical side.

The second error term in (3.2) does not exceed \( \pi \) times the number of zeros of

\[
\frac{1}{2}(F(s \pm iT) \cdot \bar{F}(s \pm iT))
\]

in the circle with center \( \sigma_0 \) and radius \( \sigma_0 - \frac{1}{2} \). Therefore, by Jensen's inequality such an error term is

\[
<< \sigma_0 \log (\max_{|s - \sigma_0| \leq \sigma_0} |F(s \pm iT)|),
\]

and hence from (3.2) we have

\[
N_F(T) = \frac{T}{\pi} \log q_F + O(\sigma_0 \log (\max_{\sigma \geq 0} |F(s)|)).
\]

Writing

\[
M = \max_{n \leq q_F} |a(n)|
\]

(and assuming that \( M \geq 2 \)) we have

\[
\max_{\sigma \geq 0} |F(s)| \ll q_F^c M,
\]

and hence (3.3) becomes

\[
N_F(T) = \frac{T}{\pi} \log q_F + O(\sigma_0 \log (q_F^c M)).
\]

Suppose now that

\[
M \ll \sigma_F e^{10\log^3 q_F}.
\]

Then (3.1) holds with the choice

\[
\sigma_0 = c \log^3 q_F
\]

for a suitable constant \( c > 0 \), and hence Proposition 1 follows immediately from (3.5)–(3.7). Therefore, in order to conclude the proof of Proposition 1 we need the following
Proposition 2. Let $F \in \mathcal{S}_0^d$ have $a(1) = 1$ and $q_F \geq 2$. Then, with the notation in (3.4), we have

$$M = O_{\sigma_F}\left(e^{10\log^3 q_F}\right).$$

We first prove a lemma. Let $\Omega(n)$ denote the total number of prime factors of $n$ and, given $\delta_0 \geq 1$, define the sequence $a(n, \delta_0)$ by induction as $a(1, \delta_0) = \delta_0$ and

$$a(n, \delta_0) = \delta_0 + \sum_{l=2}^{\Omega(n)} \sum_{n_1 \cdots n_l \geq 2} \sum_{n_1 \cdots n_l = n} a(n_1, \delta_0) \cdots a(n_l, \delta_0)$$

for $n \geq 2$, an empty sum being equal to 0. We have

Lemma 2. For $n \geq 1$

$$\delta_0 \leq a(n, \delta_0) \leq \delta_0^{\Omega(n)} 2^{\Omega(n)^2}.$$

Proof. We first note that for $l \geq 2$ and $a_1, \ldots, a_l \geq 1$ we have

$$a_1^3 + \cdots + a_l^3 \leq (a_1 + \cdots + a_l)^3 - (a_1 + \cdots + a_l)^2. \quad (3.8)$$

Indeed, (3.8) holds for $l = 2$ since $3a_1a_2^2 + 3a_1^2a_2 \geq a_1^3 + a_2^3 + 2a_1a_2 = (a_1 + a_2)^2$. Moreover, by induction we have

$$(a_1 + \cdots + a_l + a_{l+1})^3 \geq (a_1 + \cdots + a_l)^3 + a_{l+1}^3 + (a_1 + \cdots + a_l + a_{l+1})^2$$

$$\geq a_1^3 + \cdots + a_l^3 + a_{l+1}^3 + (a_1 + \cdots + a_l + a_{l+1})^2.$$

Note that the lemma is trivial when $n$ is a prime number. We prove the lemma by induction, and we may assume that $n \geq 4$ and $\Omega(n) \geq 2$. Assuming that the lemma holds for $m \leq n - 1$ and using (3.8) we have

$$\delta_0 \leq a(n, \delta_0) = \delta_0 + \sum_{l=2}^{\Omega(n)} \sum_{n_1 \cdots n_l = n} \sum_{n_1 \cdots n_l \geq 2} a(n_1, \delta_0) \cdots a(n_l, \delta_0)$$

$$\leq \delta_0 + \sum_{l=2}^{\Omega(n)} \sum_{n_1 \cdots n_l = n} \delta_0^{\Omega(n)} 2^{\Omega(n)^2} \cdots 2^{\Omega(n)^2}$$

$$\leq \delta_0 + \delta_0^{\Omega(n)} 2^{\Omega(n)^2 - \Omega(n)} \sum_{l=2}^{\Omega(n)} \sum_{n_1 \cdots n_l \geq 2} \sum_{n_1 \cdots n_l = n} 1.$$

Note that we have at most $2^{\Omega(n)}$ possible choices for each $n_j$ in the last sum, and hence

$$\sum_{l=2}^{\Omega(n)} \sum_{n_1 \cdots n_l \geq 2} 1 \leq 2^{\Omega(n)^2} \leq \sum_{l=2}^{\Omega(n)} 2^{\Omega(n)^2} \leq 2^{\Omega(n)^2} + \frac{2^{\Omega(n)^2}}{\Omega(n)}$$

$$\leq \frac{2^{\Omega(n)^2}}{2^{\Omega(n)} - 1} + \frac{2^{\Omega(n)^2}}{\Omega(n)} - 1 \leq 2^{\Omega(n)^2} - 1,$$

and the lemma follows. ■
Proof of Proposition 2. For $\sigma$ sufficiently large we can write
\[ \log F(s) = \sum_{n=2}^{\infty} b(n)n^{-s}, \tag{3.9} \]
the series being absolutely convergent. We may assume that $\sigma_F > 1$, and we first bound $\log F(s)$ for $\sigma \geq \sigma_F + \delta$, $\delta$ being a small positive constant. For $\sigma \geq \sigma_F + \frac{\delta}{2}$ we have
\[ F(s) \ll_{\delta} M, \]
and hence
\[ \Re \log F(s) = \log |F(s)| \leq c_1(\delta) \log M \]
with some $c_1(\delta) > 0$. Moreover, for every $\varepsilon > 0$ there exists $c_2(\varepsilon) > 0$ such that
\[ F(s) = 1 + O(\varepsilon) \]
for $\sigma > c_2(\varepsilon) \log M$, and hence
\[ \log F(s) = O(1). \]
Therefore, by the Borel-Carathéodory theorem we have
\[ \log F(s) = O_\delta(\log^2 M) \tag{3.10} \]
for $\sigma \geq \sigma_F + \delta$.

From (3.10) we deduce that the Lindelöf $\mu$-function of $\log F(s)$ satisfies $\mu(\sigma) = 0$ for $\sigma \geq \sigma_F$. Moreover, $\log F(s)$ is holomorphic for $\sigma > \sigma_F$. Therefore, by a general result in the theory of Dirichlet series, see chapter 9 of [7], we have that the Dirichlet series (3.9) converges for $\sigma > \sigma_F$, and hence it is absolutely convergent for $\sigma > \sigma_F + 1$. By the formula for the $n$-th coefficient of a Dirichlet series, see again chapter 9 of [7], for $\sigma > \sigma_F + 1$ we have
\[ b(n)n^{-\sigma} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log F(\sigma + it) n^{it} dt \ll \log^2 M \]
in view of (3.10). Hence
\[ |b(n)| \leq \delta_0 n^{\sigma} \log^2 M \tag{3.11} \]
for some $\delta_0 \geq 1$ and every $\sigma > \sigma_F + 1$.

Now we express the coefficients $b(n)$ in terms of the coefficients $a(n)$. For $\sigma$ sufficiently large we have
\[ F(s) = 1 + G(s) \quad \text{with} \quad |G(s)| \leq \frac{1}{2}, \]
and hence
\[
\log F(s) = \log (1 + G(s)) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} G(s)^l
= \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} \sum_n n^{-s} \left( \sum_{n_1, \ldots, n_l \geq 2} a(n_1) \cdots a(n_l) \right).
\]

Therefore, comparing Dirichlet coefficients we obtain
\[
b(n) = a(n) + \sum_{l=2}^{\Omega(n)} \frac{(-1)^{l+1}}{l} \sum_{n_1, \ldots, n_l \geq 2} a(n_1) \cdots a(n_l).
\tag{3.12}
\]

By induction, from (3.11) and (3.12) we obtain
\[
|a(n)| \leq n^{\sigma} a(n, \delta_0) \log^{2\Omega(n)} M
\tag{3.13}
\]
for \(\sigma > \sigma_F + 1\), where \(a(n, \delta_0)\) is the sequence defined before Lemma 2, starting with the \(\delta_0\) in (3.11). Indeed, for \(n = 2\) we have
\[
|a(2)| = |b(2)| \leq \delta_0 2^\sigma \log^2 M \leq 2^\sigma a(2, \delta_0) \log^{2\Omega(2)} M.
\]

Moreover, assuming (3.13) for \(2 \leq m \leq n - 1\) we get
\[
|a(n)| \leq |b(n)| + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{n_1, \ldots, n_l \geq 2} |a(n_1) \cdots a(n_l)|
\leq \delta_0 n^{\sigma} \log^2 M + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{n_1, \ldots, n_l \geq 2} a(n_1, \delta_0) \cdots a(n_l, \delta_0) n^{\sigma} \log^{2\Omega(n)} M
\leq n^{\sigma} a(n, \delta_0) \log^{2\Omega(n)} M
\]
by the inductive definition of the sequence \(a(n, \delta_0)\), and (3.13) follows. Note that (3.13) implies
\[
M \leq q_F^\sigma \max_{n | q_F} \left( a(n, \delta_0) \log^{2\Omega(n)} M \right).
\tag{3.14}
\]

Now we are ready to conclude the proof of Proposition 2. If \(M \leq \exp(\log^3 q_F)\) the result follows, and hence we may assume that \(M > \exp(\log^3 q_F)\), i.e.,
\[
\log M > \log^3 q_F.
\tag{3.15}
\]
Since $\Omega(n) \leq \frac{\log x}{\log 2}$ for $n \leq x$, from (3.14), (3.15) and Lemma 2 we have

$$M \ll q_F^\sigma (\log M)^2 \frac{\log q_F}{\log x} \frac{\log q_F}{\log M} \frac{1}{\delta_0} \frac{\log q_F}{\log x} \frac{\log q_F}{\log M} e^{4 \log^3 q_F}$$

$$\ll q_F^\sigma M^{\frac{1}{2}} \frac{\log q_F}{\log x} \frac{1}{\delta_0} \frac{\log q_F}{\log x} \frac{\log q_F}{\log M} e^{4 \log^3 q_F}$$

$$\ll q_F^\sigma M^{\frac{1}{2}} \frac{1}{\delta_0} \frac{\log q_F}{\log x} e^{4 \log^3 q_F}$$

Therefore, choosing for example $\sigma = \sigma_F + 2$ we obtain

$$M \ll q_F^\sigma \delta_0^\frac{1}{2} \frac{\log q_F}{\log x} e^{8 \log^3 q_F} \ll_{\sigma_F} c^{10 \log^3 q_F}$$

and the result follows. 

References


Addresses: Jerzy Kaczorowski, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, 60-769 Poznań, Poland;
Alberto Perelli, Dipartimento di Matematica, Via Dodecaneso 35, 16146 Genova, Italy
E-mail: kjerzy@amu.edu.pl; perelli@dima.unige.it
Received: 20 October 2003