ON SOME CONNECTIONS BETWEEN ZETA-ZEROS AND SQUARE-FREE DIVISORS OF AN INTEGER

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Abstract: A relationship between square-free divisors of an integer and zeros of the Riemann zeta-function, which is more explicit than the classical formula, is presented and discussed.

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1. Introduction and statement of results

Let \( \theta(n) \) denote the number of square-free divisors of \( n \). Moreover, let \( s(z) \) and \( S(z) \) be functions holomorphic of the upper half-plane defined by (1.1) and (1.2) below. In this note the analytic character of them is considered. In particular we show that they admit analytic continuation to multivalued functions on \( \mathbb{C} \). Moreover, \( s(z) \) satisfies certain functional equation (cf. Theorem 2 below).

In the case of simple zeros of the Riemann zeta-function, \( s(z) \) and \( S(z) \) are defined as follows:

\[
s(z) = \lim_{n \to \infty} \sum_{0 < \Re \rho < T_n} \frac{\zeta^2 \left( \frac{\rho}{2} \right) e^{2\pi z \rho}}{2 \zeta' (\rho)}
\]

and

\[
S(z) = \lim_{n \to \infty} \sum_{0 < \Re \rho < T_n} \frac{\zeta^2 \left( \frac{\rho}{2} \right) e^{2\pi z \rho}}{\rho \zeta' (\rho)},
\]

where the summation is over non-trivial zeros \( \rho \) of \( \zeta(s) \), and a suitably chosen sequence \( T_n \) yields an appropriate grouping of the zeros.

Similar investigations were performed by J. Kaczorowski [5] in connection with the distribution of primes in arithmetic progressions, and by K. Bartz [2] in connection with the Möbius \( \mu \)-function. In our case singularities of \( s(z) \) and \( S(z) \) are more complicated when compared with singularities of the corresponding functions considered in [5] and [2].

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As an application of our results we give a new proof of the classical explicit formula for \(\sum_{n \leq t} \theta(n)\).

In the general case, if \(\zeta(s)\) has a multiple zero at \(s = \rho\), the corresponding term in \(s(z)\) and \(S(z)\) must be replaced by the appropriate residue. Let \(k_\rho\) denote the multiplicity of a nontrivial zero \(\rho\). Then the general definitions of \(s(z)\) and \(S(z)\) read as follows:

\[
s(z) = \sum_{n=0}^{\infty} \sum_{T_n < \Im \rho < T_{n+1}} \frac{1}{2(k_\rho - 1)!} \frac{d^{k_\rho - 1}}{ds^{k_\rho - 1}} \left[ (s - \rho)^{k_\rho} e^{\frac{i}{2} \pi s \rho^2} \zeta^2 \left( \frac{s}{2} \right) \right]_{s=\rho}
\]

\[
S(z) = \sum_{n=0}^{\infty} \sum_{T_n < \Im \rho < T_{n+1}} \frac{1}{(k_\rho - 1)!} \frac{d^{k_\rho - 1}}{ds^{k_\rho - 1}} \left[ (s - \rho)^{k_\rho} e^{\frac{i}{2} \pi s \rho^2} \zeta^2 \left( \frac{s}{2} \right) \right]_{s=\rho}
\]

where \(\Im z > 0, T_0 = 14\), and \(2^{n-1} K_0 \leq T_n < 2^n K_0 \) (\(n \geq 1, K_0\) being an absolute positive constant) denotes a suitable sequence of numbers (for the precise definition of \(T_n\)s see Section 2). It is easy to see that \(s(z)\) and \(S(z)\) are holomorphic for \(\Im z > 0\) (see Lemma 3).

Our principal aim is to describe analytic character of these functions. To this end let us introduce the following notation. For any two real numbers \(a\) and \(b\) we denote by \(l(a, b)\) a simple and smooth curve \(\tau: [0, 1] \to \mathbb{C}\) such that \(\tau(0) = a\), \(\tau(1) = b\) and \(0 < \Im \tau(t) \leq 1\) for \(t \in (0, 1)\). Moreover,

\[
\int_{l(a,b)} f(z) \, dz
\]

for a meromorphic function \(f\) means that \(f\) is regular on the curve \(l(a, b)\) and also regular in the open domain bounded by \(l(a, b)\) and the interval \([a, b]\). Similar convention applies to integrals of type

\[
\int_{l(a,b)} f(z) \, dz.
\]

For \(z \in \mathbb{C}\) we write

\[
h(z) = \int_{l(-\frac{1}{2}, \frac{1}{2})} \frac{\zeta^2(s)}{\zeta(2s)} e^{zs} \, ds,
\]

\[
\overline{h}(z) = \int_{l(\frac{1}{2}, -\frac{1}{2})} \frac{\zeta^2(s)}{\zeta(2s)} e^{zs} \, ds.
\]

Of course \(h\) and \(\overline{h}\) are entire functions of \(z\).
Theorem 1. The function $s(z)$ is holomorphic on the upper half-plane $H = \{z \in \mathbb{C} : \Im z > 0\}$ and for $0 < \Im z < \pi$ we have

$$2\pi is(z) = \frac{i}{8} e^{-\frac{z}{2}} \sum_{m=1}^{\infty} \frac{a(m)}{m^{3/2}} \left( 1 - \frac{1}{4me^z} \right)^{-\frac{3}{2}} - \frac{e^{\frac{5z}{2}}}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{\theta(n)}{n^{3/2}(z - \log n)} + H(z) + h(z),$$

(1.5)

where $H(z)$ is holomorphic for $|\Im z| < \pi$, $h(z)$ is defined by (1.3), $a(m) = \sum_{d|m} \mu(d) \left( \frac{m}{d} \right)$ ($a(m) = 0$ iff $2^3 \parallel m$ or $3^2 \parallel m$), and the branch of the power function is chosen so that $(1 - \frac{1}{4me^z})^{-\frac{3}{2}} \to 1$ for $\Re z \to \infty$.

Let $D$ denote the complex plane with slits along half-lines $(-i\infty - \log(4m), -\log(4m))$, where $m \in \mathbb{N}$, $2^3 \not| m$ and $3^2 \not| m$.

Theorem 2. The function $s(z)$ can be continued analytically to a meromorphic function on $D$ and satisfies the following functional equation

$$s(z) + s(\overline{z}) = A(z),$$

(1.6)

where for $\Re z > -\log 4$

$$A(z) = -\frac{6}{\pi^2} e^z (z + 2\gamma - 2\zeta'(2)) + \frac{e^z}{\pi^2} \sum_{k=1}^{\infty} \frac{e^{-2kz} \zeta^2(2k)}{(4k-1)^2}. \zeta(4k-1).$$

(1.7)

The only singularities of $s(z)$ on $D$ are the simple poles at the points $z = \log n$ ($n = 1, 2, \ldots$) on the real axis with residues

$$\text{res}_{z = \log n} s(z) = \frac{\theta(n)}{2\pi i}.$$

The function $A(z)$ can be continued analytically to a multivalued analytic function on $\mathbb{C}$ except for $z = -\log 4m \pm ik\pi, m \in \mathbb{N}, 2^3 \not| m, 3^2 \not| m, k = 0, 1, 2, \ldots$, where there are polar branch points of order two.

Let us now describe analytic character of $S(z)$ using Theorems 1 and 2. It turns out that these results can be considered as a complex form of the well-known explicit formulae for $\sum_{n \leq x} \theta(n)$.

For $z \in H$ we have

$$S(z) = \int_{z \to \infty}^{z} s(u) \, du,$$

the path of integration being the half-line $u = z + iy, \infty \geq y > 0$ and $S(z)$ is defined by (1.2). Hence $S(z)$ can be continued analytically along every curve lying on $D$ and not passing through the poles of $s(z)$. $S(z)$ becomes a multivalued function on $D$. 

In fact, every pole of \( s(z) \) becomes a logarithmic branch point for \( S(z) \). In particular for \( |z - \log n| < r_0, \ n = 1, 2, \ldots, r_0 > 0 \) sufficiently small, we can write
\[
S(z) = -\frac{\theta(n)}{2\pi i} \log(z - \log n) + g(z),
\]
where \( g(z) \) is holomorphic in the disc \( |z - \log n| < r_0 \) and depends on the choice of the particular branch of \( S \).

For a real \( x \) let us write
\[
F(x) = \lim_{y \to 0^+} \Re S(x + iy). \tag{1.9}
\]
It is obvious that this limit does exist for every \( x \) which is a regular point of \( S \). For \( z = \log n, n = 1, 2, \ldots \), the limit exists as well by (1.8), since \( \lim_{y \to 0^+} \Arg(iy) = \frac{\pi}{2} \).

Furthermore, since for any \( x_0 > 0, \lim_{y \to 0^+} \Arg(x_0 + iy) = 0, \lim_{y \to 0^+} \Arg(-x_0 + iy) = -\pi \) and \( \lim_{y \to 0^+} \Arg(iy) = \frac{\pi}{2} \), we have
\[
F(x) = \frac{1}{2} (F(x + 0) + F(x - 0)) \tag{1.10}
\]
for every real \( x, x \neq -\log 4m, m \in \mathbb{N}, 2^3 \not| \ m, 3^2 \not| \ m \).

**Theorem 3.** For \( x \neq \log n, n = 1, 2, \ldots, m \in \mathbb{N}, 2^3 \not| \ m, 3^2 \not| \ m \), the series \( \sum_{k=0}^{\infty} S_k(x) \) is convergent to \( S(x) \). The convergence is uniform in every closed interval not containing points of the form \( \log n \) and \( -\log 4m \). For \( x - \log n, n = 1, 2, \ldots \), the series \( \sum_{k=0}^{\infty} \Re S_k(x) \) is convergent to \( \lim_{y \to 0^+} \Re S(x + iy) = F(x) \).

**Theorem 4.** For \( x > \frac{1}{4} \) we have
\[
\lim_{n \to \infty} \sum_{|\Omega| < T_n} \frac{1}{(k - 1)!} \frac{d^{k-1}}{ds^{k-1}} \left[ (s - \rho) \frac{\zeta^2 \left( \frac{s}{2} \right)}{\zeta(s)} x^s \right]_{s=\rho}
\]
\[
= R_0(x) - \frac{6}{x^2} \left( \log x + 2\gamma - 1 - 2\frac{\zeta'}{\zeta}(2) \right) + \frac{1}{2} - \frac{1}{\pi^2 x} \sum_{k=0}^{\infty} \frac{\zeta^2(2k + 2) x^{-2k}}{(2k + 1)} \zeta(4k + 3).
\]
where \( R_0(x) \) is defined by (2.1) below.

2. **Lemmas**

Let for \( x > 0 \)
\[
R(x) = \sum_{n \leq x} \theta(n), \quad R_0(x) = \frac{1}{2} (R(x + 0) + R(x - 0)). \tag{2.1}
\]

The symbols \( \mu(n) \) and \( d(n) \) denote as usual the Möbius function and the number of divisors of \( n \) respectively. Moreover, \( \gamma = 0.577 \ldots \) is the Euler constant.
Lemma 1 (see [4], Theorem 9.4 and [2] Lemma 1). There exist positive constants \( c_1, c_2 \) and \( t_0 \) such that for \( T \geq t_0 \), between \( T \) and \( 2T \) there exists a \( t \) satisfying

\[
|\zeta(\sigma + it)|^{-1} \leq c_2 \log^c t \quad \text{for} \quad -1 \leq \sigma \leq 3. \tag{2.2}
\]

Lemma 2. For a sufficiently small positive \( \varepsilon \) we have

\[
\zeta(\sigma + it) = \begin{cases} 
O(t^{\frac{3}{2} - \varepsilon}) & \text{for } \frac{1}{4} \leq \sigma \leq \frac{3}{4}, \\
O(t^{\frac{1}{2} - \varepsilon}) & \text{for } \frac{3}{4} \leq \sigma \leq 3.
\end{cases} \tag{2.3}
\]

For the proof see [6] and [9].

We choose \( K_0 \geq \max(t_0, 14) \) and let \( T_n(n \geq 1) \), where

\[
2^{n-1}K_0 \leq T_n < 2^nK_0
\]

is such that

\[
|\zeta(\sigma + iT_n)|^{-1} \leq c_2 \log^c T_n,
\]

(cf. (2.2)). Of course \( \zeta(s) \) has no zeros on the line \( t = T_n \). Moreover, by (2.2), (2.3) and the functional equation of \( \zeta(s) \) we have

\[
\frac{\zeta^2(\sigma + \frac{T_n}{2}i)}{\zeta(2\sigma + T_ni)} = O(T_n^{\frac{3}{2}})
\]

uniformly for \( -\frac{1}{4} \leq \sigma \leq \frac{3}{2} \).

Let us now consider uniformity of the convergence of \( s(z) \) and \( S(z) \) (see (1.1) and (1.2).

Lemma 3. The series \( s(z) \) and \( S(z) \) are uniformly convergent for \( y = \Re z \geq \delta > 0 \) almost uniformly with respect to \( z = \Re z \).

This lemma follows from (2.4). The proof is similar to the proof of Lemma 2 in [2].

Lemma 4. Let \( w_n = a_n + ib_n \), \( n = 1, 2, 3, \ldots \), denote complex numbers such that \( |a_n| \leq A, \ n \geq 1, \ b_1 < b_2 < \cdots, \lim_{n \to \infty} b_n = \infty, \ T_0 < b_1 < T_1 < T_2 < \cdots \), denote real numbers such that \( \lim_{n \to \infty} T'_n = \infty, \ h_n, \ n \geq 1 \) be the largest natural number such that \( b_{h_n} \leq T'_n \) and let \( f_n(z), \ n = 1, 2, \ldots \) be holomorphic functions for \( \Re z > -\delta, \ (\delta > 0) \). Moreover, let the series

\[
f(z) = \sum_{n=1}^{\infty} \sum_{T'_n-1 < b_k \leq T'_n} f_k(z)e^{wkz},
\]
converge for $y = \Re z > 0$ and satisfy the following two conditions

$$\left| \sum_{n=N+1}^{\infty} \sum_{T_{n-1} < b_k \leq T_n} f_k(z)e^{(w_k - w_{kN})z} \right| = o(y^{-2}), \quad N \to \infty,$$  

(2.5)

for $y \to 0^+$ almost uniformly with respect to $x = \Re z$, and

$$\left| \sum_{n=1}^{N} \sum_{T_{n-1} < b_k \leq T_n} f_k(z)e^{(w_k - w_{kN})z} \right| = o(y^{-2}), \quad N \to \infty,$$  

(2.6)

for $y \to 0^-$ also almost uniformly with respect to $x = \Re z$.

Then, if $f$ is holomorphic at the boundary point $x_0 \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \sum_{T_{n-1} \leq b_k \leq T_n} f_k(x_0)e^{u_kx_0}$$

converges to $f(x_0)$. Moreover, the convergence is uniform on every compact real interval consisting of regular points of $f$ only.

This result is a generalization of the classical theorem of M. Riesz [8]. The proof is similar to the proof of Theorem 4.2 in [5].

**Lemma 5.** Let $f$ be such as in Lemma 4 and let $b_{kN} > T_N' - C$ where $C$ is an absolute constant. Suppose that for certain $x_0 \in \mathbb{R}$ we have

$$f(z) = g \log(z - x_0) + h_1(z)$$

for $|z - x_0| < r_0$, $\Re z > 0$, where $g$ is a complex number and $h_1(z)$ is holomorphic in the whole disc $|z - x_0| < r_0$. Then for $N$ tending to infinity

$$\sum_{n=1}^{N} \sum_{T_{n-1} < b_k \leq T_n} f_k(x_0)e^{u_kx_0} = -g \log T_N - g\gamma + h_1(x_0) + \frac{\pi i}{2} + o(1).$$

The proof is similar to the proof of Theorem 4.3 in [5].

**Corollary 1.** Let $f$ be as in Lemma 5 and $\Re g = 0$. Then

$$\lim_{y \to 0^+} \Re f(x_0 + iy) = \lim_{N \to \infty} \Re \sum_{n=1}^{\infty} \sum_{T_{n-1} < b_k \leq T_n} f_k(x_0)e^{u_kx_0}.$$  

(2.7)
3. Proof of Theorem 1

Let us define the half-lines

\[ L_1: \text{the half-line: } s = -\frac{1}{4} + it, \infty > t \geq 0, \]
\[ L_2: \text{the half-line: } s = \frac{3}{2} + it, 0 \leq t < \infty, \]

\( \overline{L}_1, \overline{L}_2: \) half lines symmetrical upon the real axis to \( L_1 \) and \( L_2 \) respectively.

For \( z \in H \) we have by (2.4)

\[ 2\pi is(z) = s_1(z) + s_2(z) + h(z), \quad (3.1) \]

where

\[ s_1(z) = \int_{L_1} \frac{\zeta^2(s)}{\zeta(2s)} e^{s} ds, \quad (3.2) \]
\[ s_2(z) = \int_{L_2} \frac{\zeta^2(s)}{\zeta(2s)} e^{s} ds, \quad (3.3) \]

and \( h \) is defined by (1.3).

Let us consider \( s_1 \) first. From the functional equation for \( \zeta(s) \) we get

\[ \frac{\zeta^2(s)}{\zeta(2s)} = \frac{\sin^2 \pi s - 2 \sin^2 \frac{\pi s}{2} \Gamma^2(1-s) \Gamma(2s)}{\pi^2} \frac{\zeta^2(1-s)}{\zeta(1-2s)}. \]

Hence we can split the integral (3.2) into four integrals

\[ s_1(z) = s_{11}(z) + s_{12}(z) + s_{13}(z) + s_{14}(z), \quad (3.4) \]

where

\[ s_{11}(z) = -\frac{2}{\pi^2} \int_{L_1} \sin^2 \frac{\pi s}{2} \Gamma(1-s) \Gamma(2s) \frac{\zeta^2(1-s)}{\zeta(1-2s)} e^{s} ds, \quad (3.5) \]
\[ s_{12}(z) = -\frac{i}{4\pi^{3/2}} \int_{L_1} \Gamma(1-s) \Gamma(s + \frac{1}{2}) \frac{\zeta^2(1-s)}{\zeta(1-2s)} e^{s} ds, \]
\[ s_{13}(z) = \frac{i}{4\pi^{3/2}} \int_{L_1} \Gamma(1-s) \Gamma(s + \frac{1}{2}) \frac{\zeta^2(1-s)}{\zeta(1-2s)} e^{s} ds, \]
\[ s_{14}(z) = \frac{i}{4\pi^{3/2}} \int_{\overline{L}_1} \Gamma(1-s) \Gamma(s + \frac{1}{2}) \frac{\zeta^2(1-s)}{\zeta(1-2s)} e^{s} ds. \]

Since \( \Gamma(s) = O(|s|^{-\frac{1}{2}} \exp(-\frac{\pi}{2} |s|)) \) \( (s = \sigma + it) \), \( s_{11}(z) \) is regular for \( \Im z > -\pi \), \( s_{12}(z) \) for \( \Im z > -2\pi \), \( s_{13}(z) \) for \( \Im z < 2\pi \) and \( s_{14}(z) \) is regular for \( O < \Im z < 2\pi \).

Suppose \( z = x + iy, 0 < y < 2\pi, x > -2\log 2 \). Applying Cauchy integral theorem we obtain

\[ s_{14}(z) = \frac{1}{4\pi^{3/2} i} \cdot 2\pi i \sum_{w} \text{Res} \Gamma(1-s) \Gamma(1 + \frac{1}{2}) \frac{\zeta^2(1-s)}{\zeta(1-2s)} e^{s} ds. \]
where the summation is taken over all singularities of $\Gamma(s + \frac{1}{2})$ in the interval $(-\infty, -\frac{1}{2})$. Hence

$$s_{14}(z) = \frac{ie^{-\frac{z}{2}}}{8} \sum_{k=1}^{\infty} \frac{(2k-1)!! \cdot \zeta^2(\frac{1}{2} + k)}{8^k (k-1)!} \frac{\zeta(2k)}{\zeta(2k)} e^{-kz}$$

$$= \frac{ie^{-\frac{z}{2}}}{8} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{\mu(l)d(n)}{n^{3/2}l^2} \sum_{k=0}^{\infty} \left( -\frac{3}{2} \right)^k \left( -\frac{1}{4nl^2 e^z} \right)^k$$

$$= \frac{ie^{-\frac{z}{2}}}{8} \sum_{m=1}^{\infty} \frac{a(m)}{m^{3/2}} \left( 1 - \frac{1}{4me^z} \right)^{-\frac{3}{2}},$$

which gives analytic continuation of $s_{14}(z)$ to $z \in D$.

To compute $s_{2}(z)$ it is enough to apply the definition $\zeta(s)$ in the half-plane $\Re s > 1$. Indeed, we have

$$s_{2}(z) = \sum_{n=1}^{\infty} \frac{\theta(n)}{n^{3/2}} \int_{L_2} e^{s(z - \log n)} ds = e^{\frac{3}{2}z} \sum_{n=1}^{\infty} \frac{\theta(n)}{n^{3/2}(z - \log n)}.$$  \hspace{1cm} (3.7)

Collecting (3.1) – (3.7) we get (1.5) and Theorem 1 follows.

4. Proof of Theorem 2

Let us consider the function

$$s(z) = \lim_{n \to \infty} \sum_{|n| < \delta \rho < 0} \frac{1}{2(k_\rho - 1)!} \int_{s_\rho}^{s_0} \left[ (s - \rho) \kappa_\rho e^{\frac{1}{2}z \rho z} \zeta^2(\frac{s}{2}) \right] ds \bigg|_{s = \rho}$$

defined for $z \in \overline{H} = \{ z \in \mathbb{C} : \Re z < 0 \}$. We have

$$2\pi i s(z) = s_1(z) + \overline{s_2(z)} + \overline{h(z)},$$

where

$$s_1(z) = -\int_{L_1} \frac{\zeta^2(s)}{\zeta(2s)} e^{z\rho} ds,$$

$$s_2(z) = -\int_{L_2} \frac{\zeta^2(s)}{\zeta(2s)} e^{z\rho} ds,$$  \hspace{1cm} (4.3)

and $\overline{h}$ is defined by (1.4).
Expanding $\frac{\zeta'^2(x)}{\zeta(2s)}$ in (4.4) into Dirichlet series and using (3.7) as the definition of $\overline{s}_2(z)$ for $\Im z < 0$, it can easily be seen that
\begin{equation}
\overline{s}_2(z) = -s_2(z).
\end{equation}
Let us consider $\overline{s}_1$ next. We have
\begin{equation}
\overline{s}_1(z) = \overline{s}_{11}(z) + \overline{s}_{12}(z) + \overline{s}_{13}(z) + \overline{s}_{14}(z),
\end{equation}
where
\begin{align}
\overline{s}_{11}(z) &= \frac{2}{\pi^2} \int_{L_1} \sin^2 \frac{\pi}{2} s \Gamma^2(1-s) \Gamma(2s) \frac{\zeta'^2(1-s)}{\zeta(1-2s)} e^{\pi i s} ds, \\
\overline{s}_{12}(z) &= \frac{i}{4\pi^{3/2}} \int_{L_1} \Gamma(1-s) \Gamma \left( s + \frac{1}{2} \right) \frac{\zeta'^2(1-s)}{\zeta(1-2s)} e^{\pi i s + \pi i + \log 4} ds, \\
\overline{s}_{13}(z) &= -\frac{i}{4\pi^{3/2}} \int_{L_1} \Gamma(1-s) \Gamma \left( s + \frac{1}{2} \right) \frac{\zeta'^2(1-s)}{\zeta(1-2s)} e^{\pi i s - \pi i + \log 4} ds, \\
\overline{s}_{14}(z) &= -\frac{i}{4\pi^{3/2}} \int_{L_1-\overline{L}_1} \Gamma(1-s) \Gamma \left( s + \frac{1}{2} \right) \frac{\zeta'^2(1-s)}{\zeta(1-2s)} e^{\pi i s + \pi i + \log 4} ds
\end{align}
and $\overline{s}_{11}(z)$ is regular for $y < \pi$, $\overline{s}_{12}(z)$ for $y > -2\pi$, $\overline{s}_{13}(z)$ for $y < 2\pi$, and $\overline{s}_{14}(z)$ is regular for $-2\pi < y < 0$. Hence for $-\pi < \Im z < \pi$ we have
\begin{equation}
\overline{s}_{12}(z) = -\overline{s}_{12}(z), \overline{s}_{13}(z) = -\overline{s}_{13}(z).
\end{equation}
Similarly as before using the Cauchy integral theorem, we get for $-2\pi < \Im z < 0$, $\Re z > -2\log 2$
\begin{equation}
\overline{s}_{14}(z) = s_{14}(z).
\end{equation}
Finally for $|y| < \pi, z \in D$ by (3.1), (3.4), (4.2), (4.5), (4.6), (4.8) and (4.9) we obtain
\begin{equation}
s(z) + \overline{s}(z) = \frac{1}{2\pi i} (h(z) + \overline{h}(z)) + \frac{1}{2\pi i} (s_{11}(z) + \overline{s}_{11}(z)) + \frac{1}{\pi i} s_{14}(z).
\end{equation}
Moreover, by the theorem of residues, using (1.3) and (1.4) we have for all $z$
\begin{equation}
h(z) + \overline{h}(z) = -2\pi i \text{Res}_{s=1} \left( \frac{\zeta'^2(s)}{\zeta(2s)} e^{\pi i s} \right) = -2\pi i \frac{e^z}{\zeta(2)} (z + 2\gamma - 2\zeta'(2)).
\end{equation}
Thus for $z \in D, |y| < \pi$ by (3.6) and (4.11) we have
\begin{equation}
s(z) + \overline{s}(z) = \frac{e^z}{\zeta(2)} (z + 2\gamma - 2\zeta'(2)) + \frac{1}{2\pi i} (s_{11}(z) + \overline{s}_{11}(z))
\end{equation}
\begin{equation}
+ \frac{e^{-\frac{z}{2}}}{8\pi} \sum_{m=-\infty}^{\infty} \frac{a(m)}{m^{3/2}} \left( 1 - \frac{1}{4me^z} \right)^{-\frac{3}{2}},
\end{equation}
where $s_{11}(z) + \overline{s}_{11}(z)$ is holomorphic for $|y| < \pi$. 
Suppose \( z = x + iy, |y| < \pi, x > -2\log 2 \). Applying the Cauchy integral theorem we obtain

\[
\text{s}_{11}(z) + \overline{\text{s}}_{11}(z) = \frac{4i}{\pi} \sum_{w} \text{Res}_{w} \sin^{2} \frac{\pi}{2} s \Gamma^{2}(1 - s) \Gamma(2s) \zeta^{2}(1 - s) e^{zs},
\]

where the summation is taken over all singularities of \( \Gamma(2s) \) lying on the half-line \((-\infty, -\frac{1}{2})\). Therefore

\[
\frac{1}{2\pi i} (\text{s}_{11}(z) + \overline{\text{s}}_{11}(z)) = \frac{e^{z}}{\pi^{2}} \sum_{k=1}^{\infty} \frac{\zeta^{2}(2k)}{(4k-2) \zeta(4k-1)} \frac{e^{-2kz}}{\zeta(4k-1)} - \frac{1}{8\pi} e^{-\frac{z}{4}} \sum_{m=1}^{\infty} \frac{a(m)}{m^{3/2}} \left(1 - \frac{1}{4me^{z}}\right)^{-\frac{3}{2}}.
\]

Collecting (4.12) and (4.13) we have

\[
\text{s}(z) + \overline{\text{s}}(z) = -\frac{e^{z}}{\zeta(2)} (z + 2\gamma - 2\zeta'(2)) + \frac{e^{z}}{\pi^{2}} \sum_{k=1}^{\infty} \frac{\zeta^{2}(2k)e^{-2kz}}{(4k-2) \zeta(4k-1)}
\]

\[
= A(z).
\]

We write

\[
\sum_{k=1}^{\infty} \frac{\zeta^{2}(2k)e^{-2kz}}{(4k-2) \zeta(4k-1)} = B(z)
\]

The function \( B(z) \) is holomorphic and periodic, with period \( \pi i \) on the half-plane \( \Re z > -2\log 2 \). Hence from (4.12) and (4.14) the function \( A(z) \) can be continued analytically to a multivalued analytic function on the whole complex plane \( \mathbb{C} \) except for \( z = -\log 4m \pm ik\pi, k = 0, 1, 2, \ldots, m \in \mathbb{N}, 2^{3} \nmid m, 3^{3} \nmid m \), where there are polar branch points of order two.

If \( \rho \) is a complex zero of \( \zeta(s) \) then so is \( \overline{\rho} \). Hence for \( z \in H \) we get \( s(z) = s(\overline{z}) \). Next using (4.14) we have (1.6) and the function \( s(z) \) can be continued analytically to a meromorphic function on \( D \).

5. Proof of Theorem 3

Let us number the complex zeros of \( \zeta(s) \) lying on \( H \) according to increasing imaginary parts: \( \rho_{1}, \rho_{2}, \rho_{3}, \ldots \) and in case of equal imaginary parts according to increasing real parts.

Let \( \rho_{h_{N}} = \sigma_{h_{N}} + it_{h_{N}} \) be the last zero before the line \( T = T_{N} \).

First we verify condition (2.5) of Lemma 4. Let us define the contour \( C_{n} \) consisting of the following four parts:
By the Cauchy integral formula using estimate (2.4) we get
\[
\left| \sum_{n=N+1}^{\infty} \sum_{n_{n+1}<\rho<n_{n}} \frac{1}{(k_{\rho}-1)!} ds^{k_{\rho}-1} \left[ \frac{(s-\rho)e^{\frac{1}{2}(s-\rho_{N})\zeta^{2}(s)} + \zeta^{2}(\frac{s}{2})}{\zeta(s)} \right] \right|_{s=\rho} \leq \frac{\zeta(x)}{y} \left( \frac{1}{2N/3} + \frac{1}{y} \sum_{n=N+3}^{\infty} 2^{-\frac{1}{3}n} \right) = o_{N \to \infty} (y^{-2})
\]
for \( y \to 0^+ \) almost uniformly with respect to \( x \).

Similarly one can prove that
\[
\left| \sum_{n=1}^{N} \sum_{T_{n-1}<\rho<T_{n}} \frac{1}{(k_{\rho}-1)!} ds^{k_{\rho}-1} \left[ \frac{(s-\rho)e^{\frac{1}{2}(s-\rho_{N})\zeta^{2}(s)} + \zeta^{2}(\frac{s}{2})}{\zeta(s)} \right] \right|_{s=\rho} \leq \frac{\zeta(x)}{|y|^{2}} \left( \sum_{n=1}^{N-3} \frac{1}{2n^{-2}} + \frac{1}{2} \right) = o(|y|^{-2}) \quad (N \to \infty)
\]
for \( y \to 0^- \) almost uniformly with respect to \( x = \Re z \). Hence by Lemma 4 and Theorem 1 the series \( \sum_{n=0}^{\infty} S_n(x) \) converges to \( S(x) \) for \( x \neq \log n \), \( x \neq -\log 4m \), \( n = 1, 2, \ldots, m \in \mathbb{N}, 2^3 \nmid m, 3^3 \nmid m \).

The second part of Theorem 3 follows from (1.8) and Corollary of Lemma 5. Therefore Theorem 3 is proved.

6. Proof of Theorem 4

Suppose first that \( x > \frac{1}{4} \) and \( x \notin \mathbb{N} \), so that \( s(z) \) is regular at \( z = \log x \). Moreover, let \( -\log x \leq a < 2 \log 2 \). We have
\[
S(\log x) = S(-a) + \int_{l} s(z)dz,
\]
where \( l = l(-a, \log x) \).

By the theorem of residues and Theorem 1 we obtain
\[
\int_{l} s(z)dz - \int_{l} s(z)dz = -2\pi i \sum_{n \leq x} \text{Res}_{z=\log n} s(z) = R_0(x). \quad (6.1)
\]
Moreover, using the functional equation (1.6) we get

\begin{align*}
\int_{\log x} s(z)dz &= \int_{\log x} s(\bar{z})d\bar{z} = \int_{\log x}(-s(\bar{z}) + A(z))d\bar{z} \\
&= -\int_{\log x} s(z)dz + \int_{-a}^{\log x} A(t)dt. \tag{6.2}
\end{align*}

Combining the above equalities we arrive at

\begin{align*}
2\Re S(\log x) - 2\Re S(-a) &= R_0(x) + \int_{-a}^{\log x} A(t)dt \\
&= R_0(x) - \frac{6x}{\pi^2} (\log x - 1 + 2\gamma - 2\zeta'(2)) \\
&\quad - \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{\zeta^2(2k)x^{1-2k}}{(2k-1)(2k-2)(4k-1)} - \frac{6}{\pi^2} e^{-a}(a + 1 - 2\gamma + 2\zeta'(2)) \\
&\quad + \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{\zeta^2(2k)e^{2k-1}\gamma}{(2k-1)(4k-2k-1)(4k-1)}.
\end{align*} \tag{6.3}

Hence, by Theorem 3 we get

\begin{align*}
2F(\log x) - 2F(-a) &= R_0(x) - \frac{6x}{\pi^2} (\log x + 2\gamma - 1 - 2\zeta'(2)) \\
&\quad - \frac{6}{\pi^2} e^{-a}(a + 1 - 2\gamma + 2\zeta'(2)) \\
&\quad - \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{\zeta^2(2k)(x^{1-2k} - e^{-a(1-2k)})}{(2k-1)(4k-2k-1)(4k-1)}. \tag{6.4}
\end{align*}

Let \( N \) be a positive integer and let \( C_{N,n} \) denote the rectangle with vertices \(-N + \frac{1}{2} - i\frac{T_n}{2}, \frac{3}{2} - i\frac{T_n}{2}, \frac{3}{2} + i\frac{T_n}{2}, \) and \(-N + \frac{1}{2} + i\frac{T_n}{2}\). Then for \( \frac{1}{4} < y < 1 \), we have

\[
\lim_{n \to \infty} \int_{C_{N,n}} \frac{y^s\zeta^2(s)}{s\zeta(2s)} = 0
\]

and by the theorem of residues

\begin{align*}
\text{Res}_{s=0} \left( \frac{y^s\zeta^2(s)}{s\zeta(2s)} \right) + \text{Res}_{s=1} \left( \frac{y^s\zeta^2(s)}{s\zeta(2s)} \right) + \lim_{n \to \infty} \sum_{|\sigma|<T_n} \text{Res}_{s=\rho} \left( \frac{y^s\zeta^2(s)}{s\zeta(2s)} \right) \\
+ \lim_{n \to \infty} \sum_{k=1}^{\infty} \text{Res}_{s=-k} \frac{y^s\zeta^2(s)}{s\zeta(2s)} = 0. \tag{6.5}
\end{align*}
Therefore
\[ -\frac{1}{2} + \frac{y}{\zeta(2)} \left( \log y - 1 + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} \right) + \sum_{k=1}^{\infty} \frac{y^{1-2k} \zeta^2(2k)}{(2k-1)(2k-1)\zeta(4k-1)} + 2F(\log y) = 0. \] (6.6)

Hence, from (6.4) and (6.6) for \( y = e^{-s} \), we have (1.11).

Now, let \( x \) be a positive integer, then \( \log x \) is not a regular point of \( s(x) \).

From Theorem 3 we obtain
\[ 2F(\log x) = F(x + 0) + F(x - 0) = \frac{R(x + 0) + R(x - 0)}{2} - \frac{6x}{\pi^2} \left( \log x + 2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)} \right) + \frac{1}{2} \]
\[ - \frac{1}{\pi^2 x} \sum_{k=0}^{\infty} \frac{\zeta^2(2k+2)}{x^{2k}(2k+1)(2k+1)\zeta(4k+3)}, \]
and the proof is complete.

References


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