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RESONANCE CURVES IN THE BOMBIERI-IWANIEC METHOD

MARTIN N. HUXLEY

Abstract: The construction of resonance curves in the author's monograph 'Area, Lattice Points, and Exponential Sums' is modified so that the resonance curves now have a differential equation, a functorial mapping property, and better approximation properties.
Keywords: exponential sums, approximation, functorial.

1. Introduction

The Bombieri-Iwaniec method is used to estimate the exponential sums $S = \Sigma e(f(m))$, where $e(t) = \exp 2\pi it$ is the complex exponential function normalised to have period 1, and the phase function $f(x)$ is smooth but rapidly changing. Plots of the partial sums of S show regions of apparently random walk, and progressive spirals; Sir Michael Berry calls these spirals 'curlicues'. The curlicues occur around values of x at which $f''(x)$ takes a rational value $2a/q$ with q small. In the Bombieri-Iwaniec method the curlicues are the major arcs (regions where there is good Diophantine approximation), and the remaining regions are regarded as made up of incomplete curlicues of large radius (minor arcs). Bombieri and Iwaniec [1] were able to estimate the contribution of major arcs directly, and that of minor arcs in mean eighth power, in the special case $f(x) = T\log x$. The extension to an 'arbitrary' $f(x)$ and subsequent improvements by Huxley, Kolesnik and Watt are detailed in the monograph [3]. The method raises number-theoretic spacing problems, the second of which involves an action of the group $SL(2, Z)$ on numbers constructed from the derivatives of $f(x)$.

Iwaniec and Mozzochi [9] adapted the method to the exponential sums which arise in counting the number of integer lattice points below a smooth curve $y = g(x)$. Major arcs occur around values of x at which $g'(x)$ takes a rational value a/q with q small. The rest of the range for x is divided into minor arcs, each labelled by a rational value of $g'(x)$ on the interval, with q large. The contributions of minor arcs are estimated in mean fourth power. Iwaniec and

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Mozzochi had $g(x) = T/x$, and the present author generalised the method to 'arbitrary' $g(x)$.

In the mean fourth power argument, two different minor arcs are counted only once if there is an affine map of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{(modulo integers)},$$

with the matrix in $SL(2, Z)$, which superposes the corresponding arcs of the curve $y = g(x)$ up to a certain accuracy. The author, perhaps unwisely, called this phenomenon 'resonance'. The number of resonant pairs of minor arcs has to be estimated in order to complete the estimation of the original exponential sum. This is the only point in the Bombieri-Iwaniec and Iwaniec-Mozzochi methods where the actual form of the function $f(x)$ or $g(x)$ matters, apart from having to exclude trivial exceptional functions like $f(x) = x^2$, $g(x) = x$.

The natural method of counting would be to fix a/q, and ask how many matrices give resonances. The matrix acts on a/q by

$$\frac{a}{q} \to \frac{Aa + Bq}{Ca + Dq}.$$

This seems very difficult. Bombieri and Iwaniec fixed the matrix, and estimated the number of rationals using only two of the four coincidence conditions necessary for a resonance. The author [2] interpreted the other two conditions as: an integer point lies close to a certain curve, the 'resonance curve'. We remark at this point that the shortened version of [2] given in chapter 15 of [3] is actually wrong because the lemma corresponding to our Lemma 5.5 was omitted.

The Bombieri-Iwaniec method would reach essentially its final form if we could prove that most resonance curves have no integer point close to them. In this paper we give a precise construction of the resonance curve, involving a differential equation like that for the polar dual of a plane curve, and we prove a 'functorial under inclusion' property, which we use to obtain the relationship between integer points and resonance curves to greater accuracy. The applications to exponential sums, lattice points, and to the Riemann zeta function [4,5,6] will be published separately.

We treat the Second Spacing Problem from the beginning, with simplifications in Lemmas 4.1 and 4.7, which were the key lemmas used by Huxley and Watt in [7,8]. We do not use the results on rational points close to a curve from [2] and [3 chapter 4]. In the new approach, Lemma 4.3, that coincidences give integer points close to the resonance curve, is separated from its sharpened form Lemma 5.2, that long coincidences give integer points closer to the resonance curve. Section 5, on the correspondence between coincidences and integer points, has been expanded from the accounts in [2] and [3], which were over-simplified. Lemmas 5.4 and 5.5 should make matters clearer. The paragraph of proof corresponding to Lemma 5.5 was omitted in error from [3 chapter 15].

To sketch the functorial property, we note that the resonance curve depends both on the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $SL(2, Z)$ and on a 'reference interval' between consecutive Farey fractions $e/r < a/q < f/s$. We represent the reference interval by the matrix $\begin{pmatrix} f & e \\ s & r \end{pmatrix}$ of $SL(2, Z)$. Allowable subintervals correspond to multiplying on the right by a matrix of $SL(2, Z)$ with non-negative entries. The resonance curve for the subinterval is mapped onto the resonance curve for the whole interval, by an affine map modulo integers whose shift vector is close to an integer vector, and whose matrix is the matrix used to construct the subinterval, but transposed, because the resonance curve lies in a dual vector space. The difficult part is to show that the shift vector is close to an integer vector; approximation theory wants to be linear, but the algebra is linear fractional. It is not possible to make the shift vector zero without using the entire Taylor series of $f(x)$ or $g(x)$ in the construction. However we only require $f(x)$ to be class C^4, or $g(x)$ to be class C^3.

The precise conditions which we assume for the function $f(x)$ are as follows. We suppose that $f(x)$ is obtained on scaling a fixed function $F(x)$ by $f(x) = TF(x/M)$, which is four times continuously differentiable for $1 \leqslant x \leqslant 2$, and its derivatives satisfy:

$$|F^{(r)}(x)| \leqslant C_1 \tag{1.1}$$

for $r = 2, 3, 4$,

$$|F^{(r)}(x)| \geqslant 1/C_1 \tag{1.2}$$

for $r = 2, 3$, where C_1 is some positive constant. In some ranges we require extra conditions, either (1.2) for $r = 4$, or

$$|F''(x)F^{(4)}(x) - 3F^{(3)}(x)^2| \geqslant 1/C_3 \tag{1.3}$$

for some positive constant C_3 (the numbering of constants corresponds to Theorems 1 and 3 of [2]). We also consider a family of functions $f(x, y) = TF(x/M, y)$, which are four times partially differentiable with respect to x, with F_{11} and F_{111} non-zero twice differentiable functions of x and y, which satisfy (1.1), (1.2) and (1.3) in the appropriate ranges for each fixed y, and also

$$|(\partial_1^{r+1}F)(\partial_2\partial_1^{r+1}F) - (\partial_2\partial_1^{r}F)(\partial_1^{r+2}F)| \geqslant 1/C_4 \tag{1.4}$$

for $r = 2$, where we have written F_1 or $\partial_1 F$ for $\partial F/\partial x$, F_2 or $\partial_2 F$ for $\partial F/\partial y$, and similarly for other partial derivatives; note that F_{12} means $(F_1)_2 = \partial_2\partial_1 F$. In some ranges we require extra conditions, either (1.4) for $r = 3$, or

$$\begin{vmatrix} 3F_{111}^2 + 4F_{11}F_{1111} & 3F_{11}F_{111} & F_{11}^2 \\ F_{11111} & F_{1111} & F_{111} \\ F_{11112} & F_{1112} & F_{112} \end{vmatrix} \geqslant \frac{1}{C_5}. \tag{1.5}$$

Here again C_4 and C_5 are positive constants. The implied constants in the order of magnitude notation (Landau's O and Vinogradov's $\ll$) are constructed from the constants C_i.

This paper and [4] originally formed one long preprint, and references to [4 sections 3, 4, 5] in [5, 6] should be taken as references to this paper.

2. Subdividing the sum

We divide the sum from M to M_2 into short intervals of length N. A related parameter R is defined as the least positive integer for which $\frac{1}{2}f''(x)$ changes by at least $1/R^2$ on any interval of length N, so

$$\frac{1}{NR^2} \leqslant \frac{1}{2}\min|f^{(3)}(x)|, \qquad NR^2 \asymp \frac{M^3}{T}. \tag{2.1}$$

We label the short interval as a Farey arc $I(a/q)$. The label a/q is a rational value of $\frac{1}{2}f''(x)$, usually the rational value a/q with the smallest denominator q.

Intervals for which the smallest denominator q is too small ($q \leqslant Q_0$) or too large ($q \geqslant Q_1$) are exceptional. If the smallest denominator has $q \geqslant Q_1 \geqslant R+1$, then a/q lies between two fractions e/r and f/s in the Farey sequence $\mathcal{F}(Q_1-1)$, with $r+s \geqslant Q_1$, $1/rs > 1/R^2$. If $r \leqslant s$, then $s \geqslant Q_1/2$, and

$$r < 2R^2/Q_1, \tag{2.2}$$

$$\left|\frac{a}{q} - \frac{\mathrm{e}}{r}\right| \leqslant \frac{1}{rs} \leqslant \frac{2}{rQ_1}.$$

We take Q_0 and Q_1 satisfying $2R^2 \leqslant Q_0Q_1 \leqslant 3R^2$, so that (2.2) implies $r \leqslant Q_0$. We extend the interval $I(e/r)$ to include all x with

$$\left|\frac{\mathrm{e}}{r} - \frac{1}{2}f''(x)\right| \leqslant \frac{2}{rQ_1},$$

forming the major Farey arc $J(\mathrm{e}/r)$. A Farey arc $I(a/q)$ which does not meet any major arc now has the smallest denominator in the range $Q_0 < q < Q_1$. There are incomplete Farey arcs in the complement of the major arcs and at the ends of the interval. For technical reasons there should be at least five complete Farey arcs in each component of the remaining sum (the minor arcs). Any smaller component is divided between the major arcs on either side. After this distribution a major arc $J(\mathrm{e}/r)$ has length

$$N\left(\frac{\mathrm{e}}{r}\right) \ll NR^2\left(\frac{1}{rQ_1} + \frac{1}{R^2}\right) \ll \frac{NR^2}{rQ_1} \ll \frac{NQ_0}{r}.$$

Lemma 7.6.1 of [3] transforms the sum over a major arc into two sums of lengths at most

$$K\left(\frac{\mathrm{e}}{r}\right) \ll \frac{r}{NR^2}N^2\left(\frac{\mathrm{e}}{r}\right) \ll \frac{NQ_0^2}{rR^2}$$

plus an error term, provided that

$$Q_0 \leqslant \frac{1}{B_1}(NR^2)^{1/3}, \tag{2.3}$$

where B_1 is a constant sufficiently large in terms of the derivatives of the underlying function $F(x)$. The trivial estimate for a major arc sum is

$$\ll \sum_{k \ll K(e/r)} \sqrt{r}\left(\frac{NR^2}{kr^3}\right)^{1/4} + \frac{(NR^2)^{1/3}}{\sqrt{r}} \log M \ll$$
$$\ll \frac{NQ_0^{3/2}}{Rr} + \frac{(NR^2)^{1/3}}{\sqrt{r}} \log M.$$

The rational number e/r lies in an interval of length

$$O\left(\frac{M}{NR^2} + \frac{1}{Q_1 r}\right) = O\left(\frac{M}{NR^2}\right),$$

since $Q_1 > r$, and the choice of parameters always satisfies

$$M \gg NR. \tag{2.4}$$

Hence by [3] Lemma 7.6.1 the major arc contribution is

$$O\left(\sum_{r \leq Q_0} \frac{Mr}{NR^2}\left(\frac{NQ_0^{3/2}}{Rr} + \frac{(NR^2)^{1/3}}{\sqrt{r}} \log\, M\right)\right) =$$
$$= O\left(\frac{MQ_0^{5/2}}{R^3} + \frac{MQ_0^{3/2}}{(NR^2)^{2/3}} \log M\right) = O\left(\frac{M \log M}{\sqrt{N}}\right), \tag{2.5}$$

provided that

$$Q_0 \ll \left(\frac{R}{N}\right)^{1/5} R, \tag{2.6}$$

a condition which implies (2.3).

The remaining Farey arcs are minor arcs. On a Farey arc $\frac{1}{2}f''(x)$ runs through an interval J of length δ, where in (2.1)

$$\frac{1}{R^2} \leqslant \frac{NT}{C_1 M^3} \leqslant \delta \leqslant \frac{C_1 NT}{M^3} \leqslant \frac{2C_1^2}{R^2}. \tag{2.7}$$

Let e/r be the rational number of smallest denominator on J. If $r \geqslant R$, then we pick e/r as the rational approximation to $\frac{1}{2}f''(x)$. If $r < R$, then we pick a/q with $q > R$. The point e/r divides J into two intervals J_1 and J_2. Let J_2 be the longer, length δ_2 say. Let a/q be the rational number in J_2 with second smallest denominator. Since $(a-e)/(q-r)$ is not in J_2, we have

$$\frac{1}{qr} \leqslant \delta_2 < \frac{1}{(q-r)r},$$

so

$$q < \frac{1+\delta_2 r^2}{1+\delta_2 r},$$

and

$$\frac{1}{qr} \geqslant \frac{\delta_2}{1+\delta_2 r^2} \geqslant \frac{1/2R^2}{1+r^2/2R^2} \geqslant \frac{1}{3R^2}. \tag{2.8}$$

Having chosen the rational approximation a/q on the Farey arc, we take as centre of approximation the integer m for which $\frac{1}{2}f''(m)$ is closest to a/q, and we use the approximation for $f(m+n)$

$$f(m) + nf'(m) + \frac{an^2}{q} + \frac{n^3}{6}f^{(3)}(m),$$

writing

$$f'(m) = \frac{b+\kappa}{q}, \qquad \frac{1}{6}f^{(3)}(m) = \mu, \tag{2.9}$$

where b is an integer and κ is bounded: b is the nearest integer to $qf'(m)$ unless $2qf'(m)$ is close to an odd integer, when we consider the two nearest integers, with two choices for b and κ in (2.9). Each minor arc sum is transformed by Poisson summation (including the finite Fourier transform mod q). The Farey arc sums are grouped into classes according to the nature of the rational number a/q: the size range $Q \leqslant q \leqslant 2Q$, and, in [4], whether a/q has a good rational approximation e/r with r much smaller than q. The Poisson summation requires the parameters N and R to lie in certain ranges:

$$R \leqslant N \leqslant R^2, \qquad N^3 \ll MR^2. \tag{2.10}$$

3. Local Variables on Farey Arcs

The name 'Farey arcs' suggests a curve. The underlying curve is the graph of $y = f'(x)$. The area beneath this curve is $f(x)$, and Poisson summation interchanges the x- and y-axes. The centre of the Farey arc $I(a/q)$ is the integer m_0 for which $\frac{1}{2}f''(m_0)$ is closest to a/q. The cubic approximation to $f(x)$ on $I(a/q)$ gives a quadratic approximation to $f'(x)$:

$$y = f'(x) \cong \frac{b+\kappa}{q} + \frac{2a}{q}(x-m_0) + 3\mu(x-m_0)^2,$$

where b is an integer, $-1 < \kappa < 1$. For q odd, let e/r be the fraction before $2a/q$ in the Farey sequence $\mathcal{F}(q)$. The vectors (r, e) and $(q, 2a)$ are a basis for the integer lattice. The change of variables

$$X = ry - e(x - m_0), \qquad Y = qy - 2a(x - m_0)$$

gives

$$Y \cong b + \kappa + 3\mu q(qX - rY)^2 \cong b + \kappa + 3\mu q^3 \left(X - \frac{br}{q}\right)^2.$$

If q is even we work in $\mathcal{F}(q/2)$ and modify b and κ to get the denominator $q/2$. The position of the graph of $y = f'(x)$ with respect to the integer lattice modulo automorphisms is approximately determined by the the numbers μq^3 (real), κ (modulo one) and br/q (modulo one). In congruence notation, r is the integer $\overline{2a}$ defined by

$$1 \leqslant \overline{2a} \leqslant q, \qquad 2a.\overline{2a} \equiv 1 \pmod q.$$

The resonance curve is a device for comparing the approximations on neighbouring Farey arcs. We express the construction in terms of the function $h(v)$ defined implicitly by

$$v = \frac{1}{2}f''(x), \qquad f^{(3)}(x) = 6h(v).$$

Then

$$\frac{dv}{dx} = 3h(v), \qquad \frac{dx}{dv} = \frac{1}{3h(v)},$$

$$f^{(4)}(x) = 6h'(v)\frac{dv}{dx} = 18h(v)h'(v).$$

Lemma 3.1. (substitution) *Let m_1, $m_2 = m_1 + n$ be integers, corresponding to $v = w_1$, w_2 respectively. Let*

$$f'(m_i) = \frac{b_i + \kappa_i}{q_i},$$

$$w_i = \frac{1}{2}f''(m_i) = \frac{a_i}{q_i} + \lambda_i,$$

$$h(w_i) = \frac{1}{6}f^{(3)}(m_i) = \mu_i.$$

For $k = 0, 1, 2$ let I_k be the integrals

$$I_k = \int_{w_1}^{w_2} \left(v - \frac{a_1}{q_1}\right)^k h'(v)dv,$$

and let

$$G = \frac{1}{3\mu_2}\left(\frac{a_2}{q_2} - \frac{a_1}{q_1}\right). \tag{3.1}$$

Then

$$\frac{1}{3\mu_1} = \frac{1}{3\mu_2} + I_0, \tag{3.2}$$

$$n = G + \frac{\lambda_2}{3\mu_2} - \frac{\lambda_1}{3\mu_1} + I_1, \tag{3.3}$$

and

$$\begin{aligned}\frac{b_2+\kappa_2}{q_2} = \frac{b_1+\kappa_1}{q_1} + \frac{2a_1 n}{q_1} + 3\mu_2 G^2 + 2\lambda_2 G + \\ + \frac{\lambda_2^2 - \lambda_1^2}{3\mu_2} + I_2 + \lambda_1^2 I_0.\end{aligned} \tag{3.4}$$

Proof. We write

$$J_k = \int_{w_1}^{w_2} \frac{v^k h'(v)}{3h^2(v)} dv.$$

We obtain (3.2) at once by integration. For (3.3) we integrate by parts:

$$\begin{aligned} n &= \int_{w_1}^{w_2} \frac{1}{3h(v)} dv = \left[\frac{v}{3h(v)}\right]_{w_1}^{w_2} + \int_{w_1}^{w_2} \frac{vh'(v)}{3h^2(v)} dv = \\ &= \frac{w_2}{3h(w_2)} - \frac{w_1}{3h(w_1)} + J_1 = \\ &= \frac{1}{3\mu_2}\left(\frac{a_2}{q_2} + \lambda_2\right) - \frac{1}{3\mu_1}\left(\frac{a_1}{q_1} + \lambda_1\right) + J_1 = \\ &= \frac{\lambda_2}{3\mu_2} - \frac{\lambda_1}{3\mu_1} + \frac{1}{3\mu_2}\left(\frac{a_2}{q_2} - \frac{a_1}{q_1}\right) - \frac{a_1}{q_1} J_0 + J_1 \end{aligned}$$

by (3.2). The third term is G in (3.1), and the two integrals give I_1.

For (3.4) we start from Taylor's theorem in the form

$$f'(m+n) = f'(m) + nf''(m) + \int_m^{m+n} (m+n-x) f^{(3)}(x) dx \tag{3.5}$$

with $m = m_1$, $n = m_2 - m_1$. The integral in (3.5) becomes

$$\begin{aligned} &\int_{w=w_1}^{w_2} \int_{v=w}^{w_2} \frac{dx}{dv} dv.2dw = \int_{w=w_1}^{w_2} \int_{v=w}^{w_2} \frac{2}{3h(v)} dv dw = \\ &= \int_{w_1}^{w_2} \frac{2(v-w_1)}{3h(v)} dv = \left[\frac{(v-w_1)^2}{3h(v)}\right]_{w_1}^{w_2} + \int_{w_1}^{w_2} \frac{(v-w_1)^2 h'(v)}{3h^2(v)} dv = \\ &= \frac{(w_2-w_1)^2}{3\mu_2} + J_2 - 2w_1 J_1 + w_1^2 J_0. \end{aligned}$$

We expand

$$\frac{(w_2-w_1)^2}{3\mu_2} = 3\mu_2\left(G + \frac{\lambda_2-\lambda_1}{3\mu_2}\right)^2 = 3\mu_2 G^2 + 2(\lambda_2-\lambda_1)G + \frac{(\lambda_2-\lambda_1)^2}{3\mu_2},$$

so that (3.5) becomes

$$\frac{b_2+\kappa_2}{q_2}=\frac{b_1+\kappa_1}{q_1}+\frac{2a_1n}{q_1}+2\lambda_1 n+3\mu_2 G^2+2(\lambda_2-\lambda_1)G+ \\ +\frac{(\lambda_2-\lambda_1)^2}{3\mu_2}+I_2-2\lambda_1 J_1+\left(\frac{2a_1\lambda_1}{q_1}+\lambda_1^2\right)J_0.$$

We substitute (3.3) for $\lambda_1 n$ and cancel some terms to get (3.4). ■

In order to express Lemma 3.1 in terms of v, we write $v_i = a_i/q_i$, $\nu_i = h(v_i)$, and we define x_i by $\frac{1}{2}f''(x_i) = v_i$. By construction m_i is one of the two integers nearest to x_i (the nearest integer unless x_i is close to halfway between two consecutive integers; in counting arguments we must consider both choices for m_i in this case). We want to replace G by K where

$$K=\frac{\mu_2}{\nu_2}G=\frac{1}{3\nu_2}\left(\frac{a_2}{q_2}-\frac{a_1}{q_1}\right), \tag{3.6}$$

and I_k by J_k where

$$J_k=\int_{v_1}^{v_2}\frac{(v-v_1)^k h'(v)}{3h^2(v)}dv. \tag{3.7}$$

Lemma 3.2. (approximation) *We have*

$$\frac{\nu_i}{\mu_i}=1+O\left(\frac{1}{M}\right),\qquad \frac{K}{G}=1+O\left(\frac{1}{M}\right), \tag{3.8}$$

$$J_k=I_k+O\left(\frac{K^k}{M(NR^2)^{k-1}}\right). \tag{3.9}$$

and in (3.3) and (3.4) of Lemma 3.1

$$n=K+\frac{\lambda_2}{3\nu_2}-\frac{\lambda_1}{3\nu_1}+J_1+O\left(\frac{K}{M}\right), \tag{3.10}$$

$$\frac{b_2+\kappa_2}{q_2}=\frac{b_1+\kappa_1}{q_1}+\frac{2a_1n}{q_1}+3\nu_2K^2+2\lambda_2K+J_2+ \\ +O\left(\frac{1}{NR^2}\left(1+\frac{K^2}{M}\right)\right). \tag{3.11}$$

Proof. For (3.8) we use

$$\mu_i-\nu_i=\int_{x_i}^{m_i}\frac{1}{6}f^{(4)}(x)dx \ll \frac{1}{MNR^2}.$$

For (3.9) we use

$$I_k - J_k = \left(\int_{v_2}^{w_2} - \int_{v_1}^{w_1} \right) \frac{(v-v_1)^k h'(v)}{3h^2(v)} dv \ll$$
$$\ll \frac{1}{NR^2} \left(\frac{K}{NR^2} \right)^k \frac{(NR^2)^2}{M} \ll \frac{K^k}{M(NR^2)^{k-1}}.$$

We substitute (3.8) and (3.9) into (3.3) and (3.4). For (3.11) we have also used

$$\frac{\lambda_2^2 - \lambda_1^2}{3\mu_2} + \lambda_1^2 I_0 \ll \frac{1}{(NR^2)^2} \left(NR^2 + (w_2 - w_1) \frac{N^2R^4}{M} \right) \ll$$
$$\ll \frac{1}{NR^2} + \frac{1}{M} \cdot \frac{K}{NR^2} \ll \frac{1}{NR^2}. \qquad \blacksquare$$

To study an interval $[\alpha, \beta]$ of values of $\frac{1}{2}f''(x)$ with $\beta - \alpha < 1/3$, we consider the rational number b/j in $[2\alpha-\beta, 2\beta-\alpha]$ of least denominator. Let a/h and c/k be the predecessor and successor of b/j in the Farey sequence $\mathcal{F}(j)$. Choose integers $t \geqslant 0$, $u \geqslant 0$ with

$$\frac{a+bt}{h+jt} < 2a - \beta \leqslant \frac{a+b(t+1)}{h+j(t+1)}, \qquad \frac{b(u+1)+c}{j(u+1)+k} \leqslant 2\beta - \alpha < \frac{bu+c}{ju+k}.$$

The intervals $[(a+bt)/(h+jt), b/j]$ and $[b/j, (bu+c)/(ju+k)]$ cover the interval $[\alpha, \beta]$. If either interval does not meet $[\alpha, \beta]$, then we discard it. Each (remaining) interval is of the form $[e/r, f/s]$ with $f/s - e/r = 1/rs$, with length at least $\beta - \alpha$, with one endpoint and the mediant $(e+f)/(r+s)$ in the interval $[2\alpha - \beta, 2\beta - \alpha]$. This implies

$$(\max(r,s))^2 \geqslant \frac{1}{6(\beta-\alpha)}.$$

Each rational number in a reference interval $[e/r, f/s]$ can be written as

$$\frac{a}{q} = \frac{eu+ft}{ru+st}, \qquad (t,u) = 1.$$

This is a linear fractional map from u/t in the interval $(0, \infty)$ to a/q in the open reference interval $(e/r, f/s)$.

Lemma 3.3. (reference interval transformation) *Consider a reference interval* $e/r \leqslant a/q \leqslant f/s$ *of values of* $\frac{1}{2}f''(x)$, *with* $fr - es = 1$. *In Lemmas 3.1 and 3.2 let*

$$\frac{a_1}{q_1} = \frac{e}{r}, \qquad \frac{a_2}{q_2} = \frac{eu+ft}{ru+st} = \frac{e}{r} + \frac{t}{r(ru+st)},$$

corresponding to a change of variable from v *to* $x = u/t$ *by*

$$v = \frac{ex+f}{rx+s} = \frac{e}{r} + \frac{1}{r(rx+s)}, \qquad h(v) = h_1(x). \tag{3.12}$$

Write b, κ, λ, μ, ν *for* b_1, κ_1, λ_1, ν_1, ν_2. *In this notation (3.7) becomes*

$$J_k = -\int_{u/t}^{\infty} \frac{h_1'(x)}{3h_1^2(x)} \frac{dx}{r^k(rx+s)^k}, \tag{3.13}$$

and, corresponding to (3.1), (3.2), (3.3), and (3.4) we have

$$K = \frac{t}{3\nu r(ru+st)}, \tag{3.14}$$

$$\frac{1}{3\mu} = \frac{1}{3\nu} + J_0, \tag{3.15}$$

$$n = K + \frac{\lambda_2}{3\nu} - \frac{\lambda}{3\mu} + J_1 + O\left(\frac{K}{M}\right), \tag{3.16}$$

$$\frac{b_2+\kappa_2}{ru+st} = \frac{b+\kappa}{r} + \frac{2en}{r} + 3\nu K^2 + 2\lambda_2 K + J_2 + O\left(\frac{1}{NR^2}\left(1+\frac{K^2}{M}\right)\right). \tag{3.17}$$

Then

$$b_2 + \kappa_2 = B + \theta t + \kappa u - tG_1\left(\frac{u}{t}\right) + O\left(\frac{Q}{NR^2}\left(1+\frac{K^2}{M}\right)\right), \tag{3.18}$$

where

$$B = bu + 2(eu + ft)n \tag{3.19}$$

is an integer, θ *is a real number with*

$$\theta = \frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r}, \tag{3.20}$$

and $G_1(x)$ *is the function of* $x = u/t$ *defined by*

$$G_1(x) = \frac{1}{3h_1(x)r^2(rx+s)} + \frac{2}{r}J_1 - (rx+s)J_2, \tag{3.21}$$

with

$$G_1'(x) = -\frac{1}{3h_1(x)r(rx+s)^2} - rJ_2, \tag{3.22}$$

$$G_1''(x) = \frac{2}{3h_1(x)(rx+s)^3}. \tag{3.23}$$

Proof. We get (3.15) by direct integration, whilst (3.14), (3.16), and (3.17) are restatements of (3.6), (3.10) and (3.11), and (3.13) follows from (3.7) by the substitution (3.12). We verify (3.22) and (3.23) from the definition (3.21) of $G_1(x)$ by differentiation. The variable $x = u/t$ decreases as x increases, so there is a minus sign in (3.13) and (3.22).

For the main result (3.18) we multiply (3.17) by $ru+st$, using

$$(ru+st)\frac{e}{r} = eu + ft - \frac{t}{r}.$$

This gives

$$\begin{aligned} b_2 + \kappa_2 = bu + \kappa u + \frac{bst}{r} + \frac{\kappa st}{r} + 2(eu+ft)n - \frac{2nt}{r} + \\ + \frac{Kt}{r} + \frac{2\lambda_2 t}{3\nu r} + (ru+st)J_2 + O\left(\frac{Q}{NR^2}\left(1+\frac{K^2}{M}\right)\right), \end{aligned} \tag{3.24}$$

where we have substituted for $(ru+st)K$ from (3.14). By (3.16) we have

$$-\frac{2nt}{r} + \frac{Kt}{r} + \frac{2\lambda_2 t}{3\nu r} = -\frac{Kt}{r} + \frac{2\lambda t}{3\mu r} - \frac{2t}{r}J_1 + O\left(\frac{Kt}{Mr}\right), \tag{3.25}$$

and the error term is $O(K^2Q/MNR^2)$. We substitute (3.24) into (3.23) and collect terms to obtain (3.18). ■

A Farey arc $I(a/q)$ corresponds to a change of basis for the integer lattice to the vectors (q,a), $(\bar{a},-\bar{q})$, where $a\bar{a}+q\bar{q}=1$. If $\bar{t}$, $\bar{u}$ satisfy $t\bar{t}+u\bar{u}=1$, $\bar{u}\geqslant 1$, then we have

$$\begin{pmatrix} a & -\bar{q} \\ q & \bar{a} \end{pmatrix} = \begin{pmatrix} f & e \\ s & r \end{pmatrix}\begin{pmatrix} t & -\bar{u} \\ u & \bar{t} \end{pmatrix} = \begin{pmatrix} eu+ft & e\bar{t}-f\bar{u} \\ ru+st & r\bar{t}-s\bar{u} \end{pmatrix},$$

so that

$$(eu+ft)(r\bar{t}-s\bar{u}) - (ru+st)(e\bar{t}-f\bar{u}) = 1, \tag{3.26}$$

and

$$\frac{r}{t(ru+st)} - \frac{\bar{u}}{t} = \frac{r\bar{t}-s\bar{u}}{ru+st}. \tag{3.27}$$

Lemma 3.4. (Inverses mod q) *Write $b_2+\kappa_2 = B+L = B+\ell+\kappa_2$ in the notation (3.19), where ℓ is an integer. Let $((x))$ denote the reduction modulo 1 of x. Then*

$$\begin{aligned} \left(\left(\frac{\bar{a}b_2}{q}\right)\right) = \left(\left(\frac{(r\bar{t}-s\bar{u})b_2}{ru+st}\right)\right) = \\ = \left(\left(-\frac{1}{t}G_1'\left(\frac{u}{t}\right) - \frac{\ell\bar{u}}{t} + \frac{\kappa}{t} - \frac{\kappa_2 r}{t(ru+st)} + \frac{2\lambda_2}{3\nu(ru+st)}\right)\right) + \\ + O\left(\frac{1}{KQ} + \frac{K}{MQ}\right). \end{aligned} \tag{3.28}$$

Proof. By (3.26) we have

$$\begin{aligned} (r\bar{t}-s\bar{u})B &= (r\bar{t}-s\bar{u})bu + 2(r\bar{t}-s\bar{u})(eu+ft)n = \\ &= (r\bar{t}u + st\bar{t} - s)b + 2(1+(ru+st)(e\bar{t}-f\bar{u}))n = \\ &= 2n - bs + (ru+st)(b\bar{t} + 2(e\bar{t}-f\bar{u})n). \end{aligned}$$

Hence

$$\left(\!\!\left(\frac{(r\bar{t}-s\bar{u})B}{ru+st}\right)\!\!\right)=\left(\!\!\left(\frac{2n-bs}{ru+st}\right)\!\!\right),$$

and by (3.27)

$$\begin{aligned}\left(\!\!\left(\frac{(r\bar{t}-s\bar{u})b_2}{ru+st}\right)\!\!\right)&=\left(\!\!\left(\frac{2n-bs}{ru+st}+\frac{(r\bar{t}-s\bar{u})\ell}{ru+st}\right)\!\!\right)=\\&=\left(\!\!\left(\frac{2n-bs}{ru+st}+\left(\frac{r}{t(ru+st)}-\frac{\bar{u}}{t}\right)\ell\right)\!\!\right)=\\&=\left(\!\!\left(\frac{1}{ru+st}\left(2n-bs+\frac{rL}{t}\right)-\frac{\kappa_2 r}{t(ru+st)}-\frac{\ell\bar{u}}{t}\right)\!\!\right).\end{aligned}\tag{3.29}$$

From (3.16) and (3.18)

$$\begin{aligned}2n+\frac{rL}{t}&=2K+\frac{2\lambda_2}{3\nu}-\frac{2\lambda}{3\mu}+2J_1+O\left(\frac{K}{M}\right)+\\&\quad+\frac{r}{t}\left(\frac{(b+\kappa)st}{r}+\frac{2\lambda t}{3\mu r}+\kappa u-tG_1\left(\frac{u}{t}\right)+O\left(\frac{Q}{NR^2}\left(1+\frac{K^2}{M}\right)\right)\right)=\\&=bs+\frac{2t}{3\nu r(ru+st)}+\frac{\kappa(ru+st)}{3\nu}+\frac{2\lambda_2}{3\nu}+2J_1-\\&\quad-r\left(\frac{t}{3\nu r^2(ru+st)}+\frac{2}{r}J_1-\frac{(ru+st)}{t}J_2\right)+\\&\quad+O\left(\frac{K}{M}+\frac{NR^2}{KQ}\left(\frac{Q}{NR^2}\left(1+\frac{K^2}{M}\right)\right)\right)=\\&=bs+\frac{2\lambda_2}{3\nu}+\frac{\kappa(ru+st)}{3\nu}-\frac{ru+st}{t}G_1'\left(\frac{u}{t}\right)+O\left(\frac{1}{K}+\frac{K}{M}\right).\end{aligned}$$

We obtain (3.28) by substituting (3.30) into (3.29). ■

4. The Exact Resonance Curve

The Second Spacing Problem is to count the number of coincidences betweeen pairs of Farey arcs $I(a/q)$ and $I(a'/q')$. Let $(b+\kappa)/q$ and μ be the coefficients of the approximation on the Farey arc $I(a/q)$, and let $(b'+\kappa')/q'$, μ' be the corresponding coefficients on $I(a'/q')$. We recall that a/q and μ have fixed orders of magnitude, with

$$\frac{1}{2}f''(x)\asymp\frac{M}{NR^2},\qquad\frac{1}{6}f^{(3)}(x)\asymp\frac{1}{NR^2},\qquad f^{(4)}(x)\ll\frac{1}{MNR^2}.\tag{4.1}$$

We suppose that q and q' lie in the same range $Q\leqslant q\leqslant 2Q$. We define the inverse $\bar{a}\pmod q$ by $a\bar{a}\equiv 1\pmod q$ and similarly $a'\overline{a'}\equiv 1\pmod{q}'$. The

Coincidence Conditions can be written as

$$\left\|\frac{\overline{a'}}{q'}-\frac{\bar{a}}{q}\right\| \leqslant \Delta_1, \tag{4.2}$$

$$\left|\frac{\mu' q'^3}{\mu q^3}-1\right| \leqslant \Delta_2, \tag{4.3}$$

$$\left\|\frac{\overline{a'}b'}{q'}-\frac{\bar{a}b}{q}\right\| \leqslant \Delta_3, \tag{4.4}$$

$$|\kappa'-\kappa| \leqslant \Delta_4, \tag{4.5}$$

where $\|x\|$ denotes the absolute value of $((x))$, the reduction of x modulo one, and $\Delta_1,\ldots,\Delta_4$ are less than $\frac{1}{2}$. By (4.1), (4.3) implies $q \asymp q'$, so we lose little generality by restricting q, q' to the same range $Q \leqslant q \leqslant 2Q$. The sizes of Δ_i depend on Q. We suppose that

$$\Delta_1 \ll \frac{R^4}{HNQ^2}, \qquad \frac{1}{M} \leqslant \Delta_2 \ll \frac{R^2}{HN}, \tag{4.6}$$

$$\Delta_3 \ll \frac{R^2}{HQ}, \qquad \Delta_4 \ll \frac{Q}{H}. \tag{4.7}$$

The parameter H can be taken equal to N in this paper, but H may be smaller in other applications of resonance curves [3 chapters 8, 9].

Define the integers $\bar{q}$, $\overline{q'}$ by $a\bar{a}+q\bar{q}=1$, $a'\overline{a'}+q'\overline{q'}=1$. The First Coincidence Condition (4.2) says that there is an integer matrix of determinant one (the 'magic matrix') with

$$\begin{pmatrix} a' & -\overline{q'} \\ q' & \overline{a'} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & -\bar{q} \\ q & \bar{a} \end{pmatrix}$$

and

$$|C| = |\overline{a'}q-\bar{a}q'| \leqslant \Delta_1 qq' \leqslant 4\Delta_1 Q^2,$$

so, by (4.6), $|C|$ is bounded uniformly in Q. Since

$$\frac{q'}{q} = C\frac{a}{q}+D, \quad -\frac{q}{q'} = C\frac{a'}{q'} - A,$$

we can classify magic matrices as follows.

Type 1. The identity matrix, and a finite set of other matrices.

Type 2. Other triangular matrices with $A=D=1$, $B=0$ or $C=0$.

Type 3. Matrices with A, B, C, D non-zero, C/D negative, and

$$\frac{A}{C} \asymp -\frac{D}{C} \asymp \frac{B}{D} \asymp -\frac{B}{A} \asymp \frac{M}{NR^2}.$$

Matrices (if any) with BC non-zero that do not satisfy the conditions for type 3 must have AD bounded; they are the extra matrices of type 1.

For a type 3 matrix the fraction $v=a/q$ lies in a short interval close to $-D/C$, called the domain of the magic matrix, and $v'=a'/q'$ lies in a short interval close to A/C. Our next lemma (parts of Lemmas 14.3.1, 14.3.2 and 14.3.3 of [3]) uses the Second Coincidence Condition to shorten these intervals.

Lemma 4.1. (the Second Coincidence Condition) *For upper triangular matrices the Second Coincidence Condition* (4.3) *holds only when*

$$B \ll \frac{\Delta_2 M}{NR^2}, \tag{4.8}$$

provided that the underlying function $F(x)$ *satisfies*

$$\frac{1}{C_1} \leqslant |F^{(4)}(x)| \leqslant C_1 \tag{4.9}$$

for some constant C_1.

For lower triangular matrices the Second Coincidence Condition (4.3) *holds only when*

$$C \ll \frac{\Delta_2 NR^2}{M}, \tag{4.10}$$

provided that the underlying function $F(x)$ *satisfies*

$$\frac{1}{C_3} \leqslant |3F^{(3)}(x)^2 - F''(x)F^{(4)}(x)| \tag{4.11}$$

for some constant C_3.

For type 3 *matrices, which have*

$$C \geqslant B_2 \frac{NR^2}{M} \tag{4.12}$$

for some sufficiently large constant B_2*, the Second Coincidence Condition holds precisely when* $\frac{1}{2}f''(x)$ *lies in the intersection of the range of* $\frac{1}{2}f''(x)$ *with an interval* $D(\Delta_2)$*, the domain of the magic matrix, with length*

$$\asymp \frac{\Delta_2}{|C|}. \tag{4.13}$$

Proof. The lower bound for Δ_2 in (4.6) allows us to replace μ and μ' by $h(v)$ and $h(v')$ in (4.3) with error $O(1/M) = O(\Delta_2)$. Hence

$$\log h(v) - \log h(v') - 3\log q' + 3\log q \ll \Delta_2. \tag{4.14}$$

For upper triangular matrices $q = q'$. We have

$$\frac{d}{dv}\log h(v) = \frac{h'(v)}{h(v)} = \frac{2f^{(4)}(x)}{f^{(3)}(x)^2},$$

so by (4.1) and (4.9)

$$\left|\frac{d}{dv}\log h(v)\right| \asymp \frac{NR^2}{M}.$$

Since $B = v' - v$, we deduce (4.8).

For lower triangular matrices $a = a'$, so the left hand side of (4.14) is

$$\log h(v) - \log h(v') - 3\log v' + 3\log v.$$

We have

$$\frac{d}{d(1/v)}(\log h(v) - 3\log v) = -v^2\left(\frac{h'(v)}{h(v)} - \frac{3}{v}\right) =$$
$$= \frac{f''(x)}{2f^{(3)}(x)^2}(3f^{(3)}(x)^2 - f''(x)f^{(4)}(x)).$$

By (4.11)

$$\left|\frac{d}{d(1/v)}(\log h(v) - 3\log v)\right| \asymp \frac{M}{NR^2},$$

and since $C = 1/v' - 1/v$, we deduce (4.10).

For type 3 matrices v lies in a range of length $O(1/|C|)$ which can be covered with a bounded number of reference intervals $[e/r, f/s]$, so that v' lies in the reference interval $[e'/r', f'/s']$, where

$$\begin{pmatrix} f' & e' \\ s' & r' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} f & e \\ s & r \end{pmatrix}.$$

We extend the notation (3.12):

$$v = \frac{ex+f}{rx+s}, \quad v' = \frac{e'x+f'}{r'x+s'}, \quad h(v) = h_1(x), \quad h(v') = h_2(x), \tag{4.15}$$

so that (4.14) becomes

$$\log h_1(x) - \log h_2(x) - 3\log(r'x+s') + 3\log(rx+s) \ll \Delta_2; \tag{4.16}$$

we note that $rx + s \asymp r'x + s'$. The derivative of the left hand side of (4.16) is

$$\frac{h_1'(x)}{h_1(x)} - \frac{h_2'(x)}{h_2(x)} - \frac{3r'}{r'x+s'} + \frac{3r}{rx+s} =$$
$$= \frac{h'(v)}{h(v)}\cdot\frac{1}{(rx+s)^2} - \frac{h'(v')}{h(v')}\cdot\frac{1}{(r'x+s')^2} + \frac{3C}{(rx+s)(r'x+s')} =$$
$$= \frac{1}{(rx+s)(r'x+s')}\left(3C + O\left(\frac{NR^2}{M}\right)\right).$$

By (4.12), if the condition holds at $x = x_1$ and $x = x_2$, then

$$\int_{x_1}^{x_2} \frac{d}{dx}\log\left(\frac{h_1(x)(rx+s)^3}{h_2(x)(r'x+s')^3}\right)dx \asymp C\int_{x_1}^{x_2}\frac{dx}{(rx+s)^2} =$$
$$= C\left(\frac{1}{r(rx_1+s)} - \frac{1}{r(rx_2+s)}\right) = C\left(\frac{ex_1+f}{rx_1+s} - \frac{ex_2+f}{rx_2+s}\right) = C(v_1 - v_2).$$

The domain of the magic matrix has been restricted to the interval of v on which (4.16) holds. Its length has order of magnitude (4.13). Of course the domain may extend outside the range of v corresponding to the sum S. ∎

To express the Third and Fourth Coincidence Conditions on a reference interval we extend the notation of Lemma 3.2 and (4.15) by writing

$$J_k' = -\int_{u/t}^{\infty} \frac{h_2'(x)}{3h_2^2(x)} \frac{dx}{r'^k(r'x+s')^k}, \tag{4.17}$$

$$G_2(x) = \frac{1}{3h_2(x)r'^2(r'x+s')} + \frac{2}{r'}J_1' - (r'x+s')J_2', \tag{4.18}$$

corresponding to (3.13) and (3.21). The coincidence-detecting Fortean function is

$$g(x) = G_1(x) - G_2(x).$$

There is a technical point concerning the Fourth Coincidence Condition (4.5). We would like to relax (4.5) to

$$\|\kappa' - \kappa\| \leqslant \Delta_4. \tag{4.19}$$

We chose κ as the difference between $qf'(m)$ and the nearest integer b. Thus (4.19) implies (4.5) unless κ is close to $\pm 1/2$, when we could have κ' close to $\mp 1/2$. For κ close to $1/2$ we count the Farey arc $I(a/q)$ twice, with both choices b and $b+1$ for the nearest integer, and for κ close to $-1/2$ we count both b and $b-1$. Changing b changes the integer ℓ in Lemma 3.4. This double counting increases the number of coincidences counted, but it enables us to pass from (4.19) to (4.5) for two of the four choices of κ and κ'.

Lemma 4.2. (bounds for the Fortean function) *If the First and Second Coincidence Conditions hold at $x = u/t$, then*

$$g''(x) \ll \frac{\Delta_2 K^2 r^3}{N^2 R^4} \ll \frac{\Delta_2 N R^2}{(rx+s)^3}, \tag{4.20}$$

and for $i = 0,\ 1,\ 3$

$$g^{(i)}(x) \ll \left(\Delta_2 + \Delta_1 \frac{KQ^2}{NR^2} + \frac{K}{M}\right) \frac{K}{r} \left(\frac{Kr^2}{NR^2}\right)^i, \tag{4.21}$$

with

$$g^{(3)}(x) + \frac{3r}{rx+s} g''(x) \ll \frac{K^5 r^5}{MN^3R^6} + \Delta_1 \frac{K^5 Q^2 r^5}{N^4 R^8}. \tag{4.22}$$

Proof. From the definitions (3.7) and (4.17)

$$J_k,\ J_k' \ll \left(\frac{K}{NR^2}\right)^{k+1} \frac{N^2 R^4}{M},$$

so by (3.21), (3.22), (3.23), and (4.18)

$$
\begin{aligned}
g(x) &= \frac{1}{3h_1(x)r^2(rx+s)} - \frac{1}{3h_2(x)r'^2(r'x+s')} + O\left(\frac{K^2}{Mr}\right), \\
g'(x) &= -\frac{1}{3h_1(x)r(rx+s)^2} + \frac{1}{3h_2(x)r'(r'x+s')^2} + O\left(\frac{K^3 r}{MNR^2}\right), \\
g''(x) &= \frac{2}{3h_1(x)(rx+s)^3} - \frac{2}{3h_2(x)(r'x+s')^3}.
\end{aligned}
$$

Noting that

$$h_1'(x) = \frac{h'(v)}{(rx+s)^2} \ll \frac{1}{M(rx+s)^2},$$

and similarly for v', we have

$$g^{(3)}(x) = -\frac{2r}{h_1(x)(rx+s)^4} + \frac{2r'}{h_2(x)(r'x+s')^4} + O\left(\frac{K^5 r^5}{MN^3R^6}\right).$$

Here

$$K \asymp \frac{NR^2}{r(rx+s)} \asymp \frac{NR^2 t}{rQ}, \tag{4.23}$$

and we deduce (4.20) from the Second Coincidence Condition in the form (4.16).

We have

$$\frac{r'}{r'x+s'} - \frac{r}{rx+s} = \frac{r's - rs'}{(rx+s)(r'x+s')} = -\frac{C}{(rx+s)(r'x+s')} \ll$$

$$\ll \Delta_1 \frac{K^2 Q^2 r^2}{N^2 R^4}, \tag{4.24}$$

$$\frac{r'x+s'}{r'} - \frac{rx+s}{r} = \frac{C}{rr'} \ll \Delta_1 \frac{Q^2}{r^2}. \tag{4.25}$$

Hence

$$
\begin{aligned}
&g^{(3)}(x) + \frac{3r}{rx+s} g''(x) = \\
&= \frac{2}{h_2(x)(r'x+s')^3}\left(\frac{r'}{r'x+s'} - \frac{r}{rx+s}\right) + O\left(\frac{K^5 r^5}{MN^3R^6}\right),
\end{aligned} \tag{4.26}
$$

which gives (4.22) by (4.24). Substituting the bound (4.20) for $g''(x)$, we obtain the case $i = 3$ of (4.21). The cases $i = 0$ and $i = 1$ are proved similarly, eliminating the first term and using (4.25). ■

Lemma 4.3. (coincidence detecting) *Let* $x = u/t$, *and suppose that*

$$\frac{e}{r} + \frac{1}{2R^2} \leqslant \frac{a}{q} = \frac{eu+ft}{ru+st} \leqslant \frac{f}{s} - \frac{1}{2R^2}, \tag{4.27}$$

with $Q \leqslant q \leqslant 2Q$ *and*

$$K = \frac{t}{3h_1(u/t)r(ru+st)} \ll \sqrt{\frac{MNR^2}{H}}. \tag{4.28}$$

Let

$$y = \alpha - g'(x), \qquad z = \beta + xg'(x) - g(x), \tag{4.29}$$

where α *and* β *are the constants*

$$\alpha = \kappa - \kappa', \tag{4.30}$$

$$\beta = \frac{(b+\kappa)s}{r} - \frac{(b'+\kappa')s'}{r'} + \frac{2\lambda}{3\mu r} - \frac{2\lambda'}{3\mu' r'}. \tag{4.31}$$

Suppose that the four Coincidence Conditions (4.2) *to* (4.7) *hold on* $I(a/q)$, *with the parameter* H *in* (4.7) *and* (4.8) *satisfying*

$$H \leqslant \min(N, R^2). \tag{4.32}$$

Then there are integers c *and* d *with*

$$y = \mathrm{c} + O\left(\frac{R^2}{H(rx+s)}\right) = c + O\left(\frac{Kr}{HN}\right), \tag{4.33}$$

$$z = d + O\left(\frac{R^2 x}{H(rx+s)}\right) = d + O\left(\frac{Krx}{HN}\right). \tag{4.34}$$

Proof. The inequalities (4.27) imply

$$\frac{1}{2R^2} \leqslant \frac{t}{rq}, \qquad K \gg N, \tag{4.35}$$

$$\frac{1}{2R^2} \leqslant \frac{u}{sq}, \qquad \frac{s}{H} \leqslant \frac{R^2 u}{HQ} \ll \frac{Kru}{HNt}. \tag{4.36}$$

We use the notation of Lemmas 3.3 and 3.4, with dashes denoting the corresponding quantities on the Farey arc $I(a'/q')$. The Fourth Coincidence Condition is

$$(((\theta - \theta')t + (\kappa - \kappa')u - tg(u/t))) \ll \frac{Q}{H} + \frac{Q}{NR^2} + \frac{K^2 Q}{MNR^2}$$

by (3.18). We note that in (4.30) and (4.31) $\alpha = \kappa - \kappa'$, $\beta = \theta - \theta'$, and the error terms are all $O(Q/H)$ by (4.28) and (4.31). In the notation of Lemma 3.4 we have

$$\ell - \ell' = \beta t + \alpha u - tg\left(\frac{u}{t}\right) + O\left(\frac{Q}{H}\right). \tag{4.37}$$

The Third Coincidence Condition can be written as

$$\left(\!\!\left(-\frac{1}{t}g'\left(\frac{u}{t}\right) - \frac{\bar{u}(\ell-\ell')}{t} + \frac{\kappa-\kappa'}{t} - \frac{\kappa_2 r}{t(ru+st)} + \frac{\kappa_2' r'}{t(r'u+s't)}\right)\!\!\right) \ll$$

$$\ll \frac{R^2}{HQ} + \frac{1}{Q} + \frac{1}{KQ} + \frac{K}{MQ},$$

where we have taken the terms in λ_2 and λ_2' in (3.28) into the error term. By (4.32), (4.35), and (4.28), the error terms are all $O(R^2/HQ)$. By the identity in (4.24) we have

$$\frac{\kappa_2' r'}{t(r'u+s't)} - \frac{\kappa_2 r}{t(ru+st)} = \frac{(\kappa_2'-\kappa_2)r}{t(ru+st)} + \frac{\kappa_2'}{t}\left(\frac{r'}{r'u+s't} - \frac{r}{ru+st}\right) =$$

$$= \frac{(\kappa_2'-\kappa_2)r}{t(ru+st)} - \frac{\kappa_2' C}{(ru+st)(r'u+s't)}. \tag{4.38}$$

We use the First and Fourth Coincidence Conditions to estimate the two terms in (4.38) as

$$\ll \frac{Q}{H}\cdot\frac{r}{tQ} + \frac{R^4}{HNQ^2} \ll \frac{1}{H}\cdot\frac{NR^2}{KHQ} + \frac{R^2}{HN} \ll \frac{R^2}{HQ}.$$

The Third Coincidence Condition has been simplified to

$$\left(\!\!\left(-\frac{1}{t}g'\left(\frac{u}{t}\right) - \frac{\bar{u}(\ell-\ell')}{t} + \frac{\alpha}{t}\right)\!\!\right) \ll \frac{R^2}{HQ}. \tag{4.39}$$

Multiplying (4.39) by t, we have

$$\left(\!\!\left(\alpha - g'\left(\frac{u}{t}\right)\right)\!\!\right) \ll \frac{R^2 t}{HQ} \asymp \frac{R^2}{H(rx+s)} \asymp \frac{Kr}{HN},$$

so (4.33) holds for some integer c. Multiplying (4.39) by u, we have

$$\left(\!\!\left(\frac{\alpha u}{t} + \frac{(t\bar{t}-1)(\ell-\ell')}{t} - \frac{u}{t}g'\left(\frac{u}{t}\right)\right)\!\!\right) \ll \frac{R^2 x}{H(rx+s)}.$$

Substituting $\ell - \ell'$ from (4.36) gives

$$\left(\!\!\left(g\left(\frac{u}{t}\right) - \frac{u}{t}g'\left(\frac{u}{t}\right) - \beta\right)\!\!\right) \ll \frac{R^2 x}{H(rx+s)} + \frac{Q}{Ht} \ll$$

$$\ll \frac{Krx}{HN} + \frac{ru}{Ht} + \frac{s}{H} \ll \frac{Krx}{HN} \asymp \frac{R^2 x}{H(rx+s)}$$

by (4.35) and (4.36). Hence (4.34) holds for some integer d. ∎

The resonance curve is the locus of the point (y, z) as x varies. We summarise some useful properties of the derivatives.

Lemma 4.4. (derivatives of the resonance curve) *We have*

$$\frac{dy}{dx} = -g''(x), \qquad \frac{dz}{dx} = xg''(x),$$

and

$$\begin{aligned} \frac{dz}{dy} &= -x, & \frac{dy}{dz} &= -\frac{1}{x}, \\ \frac{d^2z}{dy^2} &= \frac{1}{g''(x)}, & \frac{d^2y}{dz^2} &= -\frac{1}{x^3 g''(x)}, \\ \frac{d^3z}{dy^3} &= \frac{g^{(3)}(x)}{g''(x)^3}, & \frac{d^3y}{dz^3} &= -\frac{3g''(x) + xg^{(3)}(x)}{x^5 g''(x)^3}. \end{aligned}$$

Proof. These formulae follow by repeated differentiation of (4.29). ■

We call the resonance curve $C(e/r, f/s; e'/r', f'/s')$; it is determined by the two reference intervals $[e/r, f/s]$ and $[e'/r', f'/s']$. Our next two lemmas show that the resonance curve depends more on the magic matrix than on the reference intervals.

Lemma 4.5. (the reverse resonance curve) *The resonance curve $C^* = C(f/s, e/r; f'/s', e'/r')$ is obtained from the resonance curve $C = C(e/r, f/s; e'/r', f'/s')$ by translation by a constant vector (η, ζ) and then interchanging the y and z axes. The vector (η, ζ) is approximately an integer vector, with*

$$\|\eta\| \ll \frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right), \qquad \|\zeta\| \ll \frac{s}{NR^2}\left(1 + \frac{K^2}{M}\right),$$

with

$$K = \frac{1}{3h_1(0)rs} \asymp NU,$$

where U is the number of Farey arcs $I(a/q)$ with $e/r < a/q \leqslant f/s$.

Proof. To adapt Lemma 4.3 to the reverse-oriented curve C^*, we replace $f(x)$ by $-f(-x)$, which changes the signs of $h(x)$, n, κ, λ, μ and the numerators e, f. At the end of the calculation we change the signs of y and z.

To compare C with C^*, we write

$$X = 1/x, \qquad h_1(x) = h_3(X), \qquad h_2(x) = h_4(X).$$

Let $j(X)$ be the analogue of $g(x)$. Then

$$j''(X) = \frac{2}{3h_3(X)(r + sX)^3} - \frac{2}{3h_4(X)(r' + s'X)^3} = x^3 g''(x).$$

The variables Y and Z on C^* have

$$Y = -\int j''(X)dX = \int xg''(x)dx,$$
$$Z = \int Xj''(X)dX = -\int g''(x)dx,$$

so that $Y = z + \zeta$, $Z = y + \eta$ for some constants η and ζ.

We use the notation of Lemma 3.3 with $a_2/q_2 = f/s$. At $x = 0$ we have $y = A - A'$, $z = B - B'$, where

$$A = \kappa - G_1'(0) = \kappa + \frac{1}{3\nu rs^2} + rJ_2,$$
$$B = \frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r} - G_1(0) =$$
$$= \frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r} - \frac{1}{3\nu r^2 s} - \frac{2}{r}J_1 + sJ_2,$$

and A', B' are the corresponding expressions for the reference interval $[e'/r', f'/s']$.

The corresponding point on C^* is $X = \infty$, where $Y = \alpha_2 = C - C'$, $Z = \beta_2 = D - D'$ with

$$C = \kappa_2, \qquad D = \frac{(b_2+\kappa_2)r}{s} - \frac{2\lambda_2}{3\nu s}$$

(the minus sign comes from the sign changes due to the orientation of C^*), and C', D' are the corresponding expressions for the reference interval $[e'/r', f'/s']$. By (3.18)

$$((\kappa_2)) = \left(\left(\frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r} - G_1(0) + O\left(\frac{s}{NR^2}\left(1 + \frac{K^2}{M}\right)\right)\right)\right),$$

so

$$\|C - B\| \ll \frac{s}{NR^2}\left(1 + \frac{K^2}{M}\right).$$

By (3.17)

$$\frac{(b_2+\kappa_2)r}{s} = b + \kappa + 2en + 3\nu K^2 r + 2\lambda_2 Kr + rJ_2 + O\left(\frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right)\right) =$$
$$= b + 2en + \kappa + \frac{1}{3\nu rs^2} + \frac{2\lambda_2}{3\nu s} + rJ_2 + O\left(\frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right)\right),$$

so

$$\|D - A\| \ll \frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right).$$

The corresponding inequalities hold for A', B', C', and D', and we deduce the Lemma. ∎

A resonance curve $C(e/r, f/s; e'/r', f'/s')$ can be constructed for any pair of reference intervals $[e/r, f/s]$ and $[e'/r', f'/s']$. It is of interest if there is a suitable magic matrix with

$$\begin{pmatrix} f' & e' \\ s' & r' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f & e \\ s & r \end{pmatrix},$$

and the Second Coincidence Condition holds at one end of the reference interval $[e/r, f/s]$. Let h, j, k, ℓ be non-negative integers with $h\ell - jk = 1$. We define

$$\begin{pmatrix} f_0 & e_0 \\ s_0 & r_0 \end{pmatrix} = \begin{pmatrix} f & e \\ s & r \end{pmatrix} \begin{pmatrix} \ell & k \\ j & h \end{pmatrix},$$

$$\begin{pmatrix} f_0' & e_0' \\ s_0' & r_0' \end{pmatrix} = \begin{pmatrix} f' & e' \\ s' & r' \end{pmatrix} \begin{pmatrix} \ell & k \\ j & h \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f_0 & e_0 \\ s_0 & r_0 \end{pmatrix}.$$

Then

$$\frac{e}{r} \leqslant \frac{e_0}{r_0} < \frac{f_0}{s_0} \leqslant \frac{f}{s}.$$

A typical rational number in the subinterval $[e_0/r_0 . f_0/s_0]$ is

$$\frac{a}{q} = \frac{e_0 u_0 + f_0 t_0}{r_0 u_0 + s_0 t_0} = \frac{eu + ft}{ru + st},$$

where

$$\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} \ell & k \\ j & h \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix}, \qquad \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} h & -k \\ -j & \ell \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix}.$$

Lemma 4.6. (affine lifting) *Let x, y, z be the variables for the resonance curve $C = C(e/r, f/s; e'/r', f'/s')$, and x_0, y_0, z_0 those for $C_0 = C(e_0/r_0, f_0/s_0; e_0'/r_0', f_0'/s_0')$. The mapping*

$$x = \frac{hx_0 + j}{kx_0 + \ell} \tag{4.40}$$

induces an affine map of C_0 into C of the form

$$(z,\ y) = (z_0, y_0) \begin{pmatrix} h & -k \\ -j & \ell \end{pmatrix} + (\zeta,\ \eta).$$

The constants η and ζ are approximately integers:

$$\eta = b_0\ell - b - 2en - b_0'\ell + b' + 2e'n' + O\left(\frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right)\right),$$

$$\zeta = -b_0 j - 2fn + b_0' j + 2f'n' + O\left(\frac{s}{NR^2} + \frac{K}{Mr}\right),$$

where $n = m_0 - m$ is the distance between the centres of approximation on the Farey arcs $I(e/r)$, $I(e_0/r_0)$, and

$$K = \frac{k}{3\mu_0 r r_0} \asymp NU$$

is the approximation to n *at* $x = h/k$. *Here* U *is the number of Farey arcs* $I(a/q)$ *with* $e/r < a/q \leqslant e_0/r_0$.

Proof. From (4.40) we have

$$rx + s = \frac{r_0 x_0 + s_0}{k x_0 + \ell}, \qquad \frac{dx}{dx_0} = \frac{1}{(k x_0 + \ell)^2}.$$

Let

$$h_3(x) = h\left(\frac{e_0 x + f_0}{r_0 x + s_0}\right), \qquad h_4(x) = h\left(\frac{e_0' x + f_0'}{r_0' x + s_0'}\right).$$

Let $g_0(x_0)$ be the Fortean function in the construction of C_0. Then

$$g_0''(x_0) = \frac{2}{3h_3(x_0)(r_0 x_0 + s_o)^3} - \frac{2}{3h_4(x_0)(r_0' x_0 + s_0')^3} = \frac{g''(x)}{(k x_0 + \ell)^3}.$$

Then

$$y = -\int g''(x)dx = \int (k x_0 + \ell) g_0''(x_0) dx_0 = -k z_0 + \ell y_0 + \eta,$$

$$z = \int x g''(x) dx = \int (h x_0 + j) g_0''(x_0) dx_0 = h z_0 - j y_0 + \zeta$$

for some constants η, ζ.

We compute the constants by putting $x = h/k$, $x_0 = \infty$, so $rx + s = r_0/k$. We have

$$y = \alpha - g'(x) = A - A',$$

where by (3.22)

$$A = \kappa - G_1'\left(\frac{h}{k}\right) = \kappa + \frac{k^2}{3\mu_0 r r_0^2} + r J_2,$$

and

$$z = \beta + x g'(x) - g(x) = B - B',$$

where by (3.21) and (3.22)

$$B = \frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r} + \frac{h}{k} G_1'\left(\frac{h}{k}\right) - G_1\left(\frac{h}{k}\right) =$$

$$= \frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r} - \frac{(hr + r_0)k}{3\mu_0 r^2 r_o^2} - \frac{2}{r} J_1 + s J_2.$$

At $x_0 = \infty$ on C_0 we have

$$\ell \alpha_0 - k \beta_0 = C - C', \qquad -j \alpha_0 + h \beta_0 = D - D',$$

where α_0 and β_0 are given by (4.31) for the curve C_0, so

$$C = -\frac{k(b_0 + \kappa_0)s_0}{r_0} - \frac{2k\lambda_0}{3\mu_0 r_0} + \ell \kappa_0 = -b_0 \ell + \frac{r}{r_0}(b_0 + \kappa_0) - 2Kr\lambda_0,$$

$$D = \frac{h(b_0 + \kappa_0)s_0}{r_0} + \frac{2h\lambda_0}{3\mu_0 r_0} - j\kappa_0 = b_0 j + \frac{s}{r_0}(b_0 + \kappa_0) + \frac{2hKr\lambda_0}{k}.$$

Substituting from (3.17), we have

$$\begin{aligned}C &= -b_0\ell - 2Kr\lambda_0 + b + \kappa + 2en + 3\mu_0 K^2 r + 2\lambda_0 Kr + \\ &+ rJ_2 + O\left(\frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right)\right) = \\ &= -b_0\ell + b + 2en + A + O\left(\frac{r}{NR^2}\left(1 + \frac{K^2}{M}\right)\right).\end{aligned}$$

Similarly by (3.17) and (3.16)

$$\begin{aligned}D &= b_0 j + \frac{2hKr\lambda_0}{k} + \frac{(b+\kappa)s}{r} + \frac{2ens}{r} + 3\mu_0 K^2 s + 2\lambda_0 Ks + \\ &+ sJ_2 + O\left(\frac{s}{NR^2}\left(1 + \frac{K^2}{M}\right)\right) = \\ &= b_0 j + 2fn - \frac{2n}{r} + \frac{(b+\kappa)s}{r} + \frac{2\lambda_0 Kr_0}{k} + \\ &+ \frac{k^2 s}{3\mu_0 r^2 r_0^2} + sJ_2 + O\left(\frac{s}{NR^2}\left(1 + \frac{K^2}{M}\right)\right) = \\ &= b_0 j + 2fn + \frac{(b+\kappa)s}{r} + \frac{2\lambda}{3\mu r} - \frac{2J_1}{r} + O\left(\frac{K}{Mr}\right) - \\ &- \frac{k(r_0 + hr)}{3\mu_0 r^2 r_0^2} + sJ_2 + O\left(\frac{s}{NR^2}\left(1 + \frac{K^2}{M}\right)\right) = \\ &= b_0 j + 2fn + B + O\left(\frac{K}{Mr} + \frac{s}{NR^2} + \frac{K^2 s}{MNR^2}\right).\end{aligned}$$

In the error term we have

$$\frac{s}{NR^2} \ll \frac{r_0}{kNR^2} \asymp \frac{1}{Kr},$$

so we may drop the last term. ■

The conclusions of Lemma 4.6 remain true if we allow $f = 1$, $s = 0$ or $e = -1$, $r = 0$, so that the label a/q is infinite at the end of the resonance curve $C(e/r, f/s; e'/r', f'/s')$.

It is also possible to sharpen Lemma 4.6 as follows. An integer point (c_0, d_0) in a disc with centre (y_0, z_0) on $C_0 = C(e_0/r_0, f_0/s_0; e_0'/r_0', f_0'/s_0')$ is lifted to an integer point (c, d) in an ellipse with centre (y, z) on $C(e/r, f/s; e'/r', f'/s')$. The major axis of the ellipse is close to the tangent at (y, z), so the nearest point on the resonance curve to (c, d) is much closer than (y, z). There are exceptional cases when (y, z) is near the cusp or an end of the curve $C(e/r, f/s; e'/r', f'/s')$. We do not need to use this refinement in [4].

In Proposition 2 we have a family of sums with different values of a parameter y with $0 \leqslant y \leqslant 1$. We write the Farey arcs as $I(a/q, y)$ to indicate which sum of the family they come from. The underlying function $F(x,y)$ depends on x and y. We write F_1, f_1 for $\partial F/\partial x$, $\partial f/\partial x$, F_2 and f_2 for $\partial F/\partial y$, $\partial f/\partial y$, and similarly for higher partial derivatives. The function $h(v,w)$ depends on v and w. To avoid a clash of notation we use ∂_1 for $\partial/\partial v$ and ∂_2 for $\partial/\partial w$ to indicate the partial derivatives of $h(v,w)$. Although w is the same variable as y, we have

$$\frac{\partial x}{\partial v} = \frac{2}{f_{111}}, \quad \frac{\partial x}{\partial w} = -\frac{f_{112}}{f_{111}}, \quad \frac{\partial y}{\partial v} = 0, \quad \frac{\partial y}{\partial w} = 1,$$

so the Jacobian matrix has terms off the diagonal.

Lemma 4.7. (the Second Coincidence Condition for a family of sums) *Suppose that $F(x,y)$ is defined for $1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$, and four times partially differentiable with respect to x, and that $F_{11}(x,y)$, $F_{111}(x.y)$ are non-zero differentiable functions of x and y with*

$$|F_{111}F_{1112} - F_{112}F_{1111}| \geqslant 1/C_5 \tag{4.41}$$

for some constant C_5. Then for fixed a/q, a'/q' and y, the values of y' for which the Farey arcs $I(a/q,y)$, $I(a'/q',y')$ satisfy the Second Coincidence Condition lie in an interval of length $O(\Delta_2)$.

Let $J(y)$ be the range of $v = \frac{1}{2}f_{11}(x,y)$ as x varies with y fixed. For fixed y and y' and a fixed type 3 magic matrix, the Second Coincidence Condition holds if v lies in the intersection of $J(y)$ with an interval $D = D(\Delta_2, y, y')$, the domain of the magic matrix. The length of D is

$$\asymp \Delta_2/|C|. \tag{4.42}$$

For fixed y and y' and fixed upper triangular matrix $\begin{pmatrix}1 & B\\ 0 & 1\end{pmatrix}$, the Farey arcs $I\left(\frac{a}{q}, y\right)$ and $I\left(\frac{a}{q} + B, y'\right)$ which coincide in the Second Coincidence Condition have

$$B \ll (|y-y'| + \Delta_2)\frac{M}{NR^2}. \tag{4.43}$$

For

$$B_3\Delta_2 \leqslant |y-y'| \leqslant 1/B_3, \tag{4.44}$$

where the constant B_3 is sufficiently large in terms of the range of the derivatives of $F(x,y)$, the rational a/q must lie in the intersection of $J(y)$ with an interval $D = D(\Delta_2, y, y')$, provided that F_{11} and F_{111} are twice differentiable functions of x and y, satisfying (4.9) with

$$|F_{1111}F_{11112} - F_{1112}F_{11111}| \geqslant 1/C_4. \tag{4.45}$$

The length of the domain D is

$$\asymp \frac{\Delta_2 M}{NR^2|y-y'|}. \tag{4.46}$$

For fixed y and y' and fixed lower triangular matrix $\left(\begin{smallmatrix}1 & 0\\ C & 1\end{smallmatrix}\right)$, the Farey arcs $I\left(\frac{a}{q}, y\right)$ and $I\left(\frac{a}{q+aC}, y'\right)$ which coincide in the Second Coincidence Condition have

$$C \ll (|y-y'| + \Delta_2)\frac{NR^2}{M}. \tag{4.47}$$

When (4.44) holds, the rational a/q must lie in the intersection of $J(y)$ with an interval $D = D(\Delta_2, y, y')$, provided that F_{11} and F_{111} are twice differentiable functions of x and y, satisfying (4.11), with

$$|E| \geqslant 1/C_5, \tag{4.48}$$

where E is the determinant

$$E = \begin{vmatrix} 3F_{111}^2 + 4F_{11}F_{1111} & 3F_{11}F_{111} & F_{11}^2 \\ F_{11111} & F_{1111} & F_{111} \\ F_{11112} & F_{1112} & F_{112} \end{vmatrix}.$$

The length of the domain D is given by (4.46).

The implied constants are constructed from $C_1, \ldots, C_5$ and from the upper bounds for the partial derivatives of $F(x, y)$.

Proof. The condition (4.14) becomes

$$\log h(v, y) - \log h(v', y') + 3\log q - 3\log q' \ll \Delta_2. \tag{4.49}$$

If v, v', and y are fixed, then the only variable term in (4.49) is $\log h(v', y')$. We have

$$\partial_2 \log h(v, y) = \frac{\partial_2 h(v, y)}{h(v, y)} = \frac{f_{111}f_{1112} - f_{112}f_{1111}}{f_{111}^2},$$

which is uniformly bounded away from zero by (4.41). Hence for fixed v, v', y, the parameter y' lies in an interval of length $O(\Delta_2)$.

In the other assertions of the Lemma, y, y' and the magic matrix are fixed. For type 3 magic matrices the argument of Lemma 4.1 works the same, irrespective of the values of the parameters.

For upper triangular matrices we have $v' = v + B$, $q' = q$. We put $y' = y + \eta$. Then (4.49) becomes

$$\log h(v, y) - \log h(v + B, y + \eta) \ll \Delta_2.$$

By the mean value theorem, there is a τ between 0 and 1 with

$$B\partial_1 h(v + \tau B, y + \tau\eta) + \eta\partial_2 h(v + \tau B, y + \tau\eta) \ll \Delta_2 \max|h| \ll \frac{\Delta_2}{NR^2}. \tag{4.50}$$

We have

$$|\partial_1 h| \asymp 1/M, \qquad |\partial_2 h| \ll 1/NR^2,$$

For lower triangular matrices $a = a'$, so the left hand side of (4.49) is

$$\log h(v,y) - \log h(v',y') - 3\log v + 3\log v'.$$

We put $u = 1/v$,

$$H(x,y) = u^3 h(1/u, y),$$

and we write $\partial_1 H$ for $\partial H/\partial u$, et cetera. We follow the same argument as above, using (4.11) and (4.48) in place of (4.9) and (4.45) to find that u must lie in an interval of length

$$\asymp \frac{\Delta_2 N R^2}{M|y-y'|},$$

which corresponds to an interval for v of length (4.46). ∎

5. Coincident Farey Arcs and Integer Points

First we compare the approximations on neighbouring Farey arcs.

Lemma 5.1. (neighbouring rational approximations) *Suppose that the Coincidence Conditions* (4.2) *to* (4.5) *hold on the Farey arc* $I(e/r)$, *so that*

$$C \ll \frac{R^4}{HN}, \tag{5.1}$$

$$\frac{\mu' r'^3}{\mu r^3} - 1 \ll \frac{R^2}{HN}, \tag{5.2}$$

$$\left\| \frac{bs}{r} - \frac{b's'}{r'} \right\| \ll \frac{R^2}{Hr}, \tag{5.3}$$

$$|\kappa - \kappa'| \ll \frac{r}{H}, \tag{5.4}$$

with

$$H \leqslant \min\left(N,\, R^2,\, \frac{MR^2}{N^2}\right). \tag{5.5}$$

Suppose that a/q *lies in the reference interval* $[e/r,\ f/s]$ *with* $Q \leqslant q \leqslant 2Q$, *and*

$$\frac{B_4 R^2}{H} \leqslant r \leqslant Q \leqslant \frac{H}{B_4}, \tag{5.6}$$

with B_4 *sufficiently large in terms of the derivatives of the underlying function* $F(x)$, *and that*

$$\frac{e}{r} + \frac{1}{4R^2} \leqslant \frac{a}{q} \leqslant \frac{e}{r} + \frac{10C_1^2}{R^2}. \tag{5.7}$$

$$|\partial_1^2 h| \ll \frac{NR^2}{M^2}, \qquad |\partial_1\partial_2 h| \ll \frac{1}{M}, \qquad |\partial_2^2 h| \ll \frac{1}{NR^2}.$$

Hence

$$B \ll (|\eta| + \Delta_2)\,\frac{M}{NR^2}.$$

Now

$$\begin{aligned}&\frac{d}{dt}\left(B\partial_1 h(v+\tau B, y+\tau\eta) + \eta\partial_2 h(v+\tau B, y+\tau\eta)\right) = \\ &= \left(B^2\partial_1^2 + 2B\eta\partial_1\partial_2 + \eta^2\partial_2^2\right) h(v+\tau B, y+\tau\eta) \ll \\ &\ll \left(\frac{|B|NR^2}{M} + |\eta|\right)^2 \frac{1}{NR^2} \ll (\eta^2 + \Delta_2^2)\frac{1}{NR^2}.\end{aligned}$$

Hence we have for $0 \leqslant t \leqslant 1$

$$B\partial_1 h(v+\tau B, y+\tau\eta) + \eta\partial_2 h(v+\tau B, y+\tau\eta) \ll (\eta^2 + \Delta_2)\frac{1}{NR^2}.$$

The condition (4.41) implies

$$|(\partial_1 h)(\partial_1\partial_2 h) - (\partial_2 h)(\partial_1^2 h)| \geqslant \frac{1}{9C_1^2 C_5 M^2}.$$

Now

$$\begin{aligned}&(\partial_1 h)(B\partial_1^2 h + \eta\partial_1\partial_2 h) = \\ &= \eta(\partial_1 h)(\partial_1\partial_2 h) - \eta(\partial_2 h)(\partial_1^2 h) + O\left(\frac{|\partial_1^2 h|}{NR^2}(\eta^2 + \Delta_2)\right) = \\ &= \eta\left((\partial_1 h)(\partial_1\partial_2 h) - (\partial_2 h)(\partial_1^2 h)\right) + O\left(\frac{\eta^2 + \Delta_2}{M^2}\right).\end{aligned}$$

Since η satisfies (4.44) with B_3 sufficiently large, we have

$$|B\partial_1^2 h + \eta\partial_1\partial_2 h| \asymp \frac{|\eta|}{M}.$$

By continuity $B\partial_1^2 h + \eta\partial_1\partial_2 h$ has constant sign, so

$$\left|\frac{\partial}{\partial v}\left(h(v+B, y+\eta) - h(v,y)\right)\right| =$$

$$= \left|\int_0^1 \left(B\partial_1^2 h(v+\tau B, y+\tau\eta) + \eta\partial_1\partial_2 h(v+\tau B, y+\tau\eta)\right) dt\right| \asymp \frac{|\eta|}{M}.$$

Hence

$$\left|\frac{\partial}{\partial v}\left(\log h(v+B, y+\eta) - \log h(v,y)\right)\right| \asymp \frac{|\eta| NR^2}{M^2},$$

and (4.49), if it holds anywhere, holds for an interval in v of length given by (4.43).

Then the four Coincidence Conditions hold for $I(a/q)$ to the accuracy of (4.6) *and* (4.7) *weakened by a bounded factor. In Lemma* 4.3 *the points* (α, β) *corresponding to* $I(e/r)$ *and* (y, z) *corresponding to* $I(a/q)$ *are close to the same integer point* (c, d).

Proof. Since the magic matrix is fixed, the First Coincidence Condition (5.1) still holds on $I(a/q)$. By Lemma 4.1, the Second Coincidence Condition holds for an interval $D(\Delta_2)$ in v of length

$$\asymp \frac{\Delta_2}{|C|} \gg \frac{1}{R^2}$$

by (5.1) and (5.2); if we multiply Δ_2 by a factor, then the interval $D(\Delta_2)$ extends proportionately. Hence the Second Coincidence Condition holds at $v = a/q$, $x = u/t$, weakened by a bounded factor.

In the notation (3.14), we have by (5.7)

$$K = \frac{t}{3\nu r(ru + st)} \asymp N, \qquad \frac{t}{r} \asymp \frac{Q}{R^2}. \tag{5.8}$$

We write (5.3) as

$$\frac{bs}{r} - \frac{b's'}{r'} = h + \gamma, \qquad |\gamma| \ll \frac{R^2}{Hr}. \tag{5.9}$$

From (3.18), in the notation of Lemma 3.4,

$$((\kappa_2 - \kappa_2')) = ((L - L'))$$

with

$$\begin{aligned} L - L' &= \\ &= (\theta - \theta')t + (\kappa - \kappa')u - tg\left(\frac{u}{t}\right) + O\left(\frac{Q}{NR^2}\left(1 + \frac{K^2}{M}\right)\right) = \\ &= ht + \gamma t + O\left(\frac{t}{r}\right) + \frac{(\kappa - \kappa')(ru + st)}{r} - tg\left(\frac{u}{t}\right) + O\left(\frac{Q}{NR^2}\left(1 + \frac{R^2}{H}\right)\right) = \\ &= ht + O\left(\frac{R^2 t}{Hr} + \frac{Q}{H} + \left(\frac{R^2}{HN} + \frac{N}{M}\right)\frac{Nt}{r} + \frac{Q}{NR^2} + \frac{Q}{HN}\right) \\ &= ht + O\left(\frac{Q}{H}\right), \end{aligned} \tag{5.10}$$

where we have used (5.8), (5.9), (5.5), (5.4), and (4.21). The term $O(Q/H)$ is numerically less than $1/2$ if B_4 is sufficiently large in (5.6), so, in the notation of Lemma 3.4, $\ell - \ell' = ht$. In the Third Coincidence Condition we have by (3.28) and (4.37)

$$\left(\!\!\left(\frac{\bar{a}b_2}{q} - \frac{\overline{a'}b_2'}{q'}\right)\!\!\right) =$$

$$= \left(\!\!\left(-\frac{1}{t}g'\left(\frac{u}{t}\right) - \frac{\bar{u}(\ell - \ell')}{t} + \frac{\kappa - \kappa'}{t} - \frac{\kappa_2 r}{t(ru+st)} + \frac{\kappa_2' r'}{t(r'u+s't)}\right)\!\!\right) +$$

$$+ O\left(\frac{1}{Q} + \frac{1}{KQ} + \frac{K}{MQ}\right) =$$

$$= \left(\!\!\left(-\frac{1}{t}g'\left(\frac{u}{t}\right) - h\bar{u} + \frac{\kappa - \kappa'}{t} + \frac{(\kappa_2' - \kappa_2)r}{t(ru+st)} - \frac{\kappa_2' C}{(ru+st)(r'u+s't)}\right)\!\!\right) +$$

$$+ O\left(\frac{1}{Q} + \frac{1}{NQ} + \frac{R^2}{HQ}\right) \ll$$

$$\ll \frac{1}{t}\left(\frac{R^2}{HN} + \frac{N}{M}\right)\frac{Nr}{R^2} + \frac{r}{Ht} + \frac{Q}{H}\cdot\frac{r}{Qt} + \frac{R^4}{HNQ^2} + \frac{R^2}{HQ} \ll \frac{R^2}{HQ},$$

where we have used (5.8), (4.21), (5.5), (5.4), (5.10), (5.1), and (5.6). Hence the Third and Fourth Coincidence Conditions hold on $I(a/q)$ with accuracy (4.7), but the constants implied in the $\ll$ sign are larger than in (5.3) and (5.4).

For each $x = u/t$ corresponding to a/q in the range (5.7), the point (y, z) is close to some integer point (c, d) by Lemma 4.3. The endpoint (α, β) is close to $(0, h)$. As in the previous calculation, (4.21) gives

$$g'\left(\frac{u}{t}\right) \ll \frac{r}{H} \ll \frac{1}{B_4}$$

by (5.6), so $c = 0$ if B_4 is sufficiently large. Hence

$$\frac{u}{t}g'\left(\frac{u}{t}\right) \ll \frac{Q}{rt}\cdot\frac{r}{H} \ll \frac{1}{B_4},$$

and as in (5.10)

$$g\left(\frac{u}{t}\right) \ll \frac{R^2}{Hr} \ll \frac{1}{B_4},$$

so $d = h$ if B_4 is sufficiently large in (5.6). This completes the proof of the Lemma. ■

In section 2 there was a case when the rational number e/r of smallest denominator had $r < R$, and we picked another label a/q, with

$$\frac{1}{3R^2} \leqslant \left|\frac{a}{q} - \frac{e}{r}\right| \leqslant \frac{2C_1}{R^2}$$

by (2.7) and (2.8). Taking e/r and f/s of Lemma 5.1 to be the e/r and a/q of section 2, we see that the Coincidence Conditions hold at e/r if and only if they hold at a/q (up to a bounded factor in the inequalities). Hence the change of label does not essentially affect whether a coincidence occurs.

Lemma 5.2. (coincident consecutive Farey arcs) *Suppose that the conditions of Lemma* 5.1 *hold, and that there are at least* $L-2$ *Farey arcs strictly between* $I(e/r)$ *and* $I(f/s)$, *and that the* $L-1$ *Farey arcs* $I(a/q)$ *following* $I(e/r)$ *have* q *in a range* $Q \leqslant q \leqslant 2Q$ *satisfying* (5.6), *and that the four Coincidence Conditions* (4.2) *to* (4.5) *hold with the appropriate accuracy* (4.6) *and* (4.7). *Suppose also that*

$$L \ll \sqrt{\frac{MR^2}{HN}}. \tag{5.11}$$

Then the Second Coincidence Condition (4.3) *holds with*

$$\Delta_2 \ll \frac{R^2}{HL^2N}, \tag{5.12}$$

and if the magic matrix is type 3, *then*

$$C \ll \frac{R^4}{HL^3N}. \tag{5.13}$$

The points (y,z) *on the resonance curve corresponding to the* L *Farey arcs are close to the same integer point* (c,d), *and there is an intermediate value of* x *for which*

$$y = c + O\left(\frac{r}{H}\right) = c + O\left(\frac{R^2}{HL(rx+s)}\right), \tag{5.14}$$

$$z = d + O\left(\frac{rx+s}{H}\right) = d + O\left(\frac{R^2}{HLr}\right), \tag{5.15}$$

and

$$K = \frac{1}{3h_1(x)r(rx+s)} \asymp LN. \tag{5.16}$$

Proof. We number the L Farey arcs as $I(a_i/q_i)$, where $a_0/q_0 = e/r$, and

$$v_i = \frac{a_i}{q_i} = \frac{eu_i + ft_i}{ru_i + st_i}, \qquad x_i = \frac{u_i}{t_i}, \qquad rx_i + s_i = \frac{q_i}{t_i}.$$

Then

$$v_{i+2} - v_i \asymp 1/R^2, \tag{5.17}$$

for $i = 0, \ldots, L-3$. The Fourth Coincidence Condition is

$$\left(\!\!\left(\beta t_i + \alpha u_i - t_i g\left(\frac{u_i}{t_i}\right)\right)\!\!\right) \ll \frac{q_i}{H}.$$

From (5.4) of Lemma 5.1

$$\alpha u_i \ll \frac{ru_i}{H} \ll \frac{q_i}{H},$$

so in the notation (5.9) we have

$$\left(\left(\gamma t_i - t_i g\left(\frac{u_i}{t_i}\right)\right)\right) \ll \frac{q_i}{H}. \tag{5.18}$$

The bound (5.10) in the proof of Lemma 5.1 says

$$\gamma t_i - t_i g\left(\frac{u_i}{t_i}\right) \ll \frac{q_i}{H} \tag{5.19}$$

when (5.7) holds, so (5.19) is true for $i = 1, 2, 3, 4$.

We sharpen (5.18) to (5.19) for $i = 5, \ldots, L-1$ by induction on i. Suppose that $k \geqslant 4$ and (5.19) holds for $i = 1, \ldots, k$. Since $k - 2 > 2$, we have

$$rx + s \asymp rx_{k+1} + s$$

for $x_{k-2} \leqslant x \leqslant x_{k+1}$, and so for $j = k-2$ or k we have

$$x_j - x_{k+1} = -\int_{v_{k+1}}^{j} (rx+s)^2 dv \ll \frac{(rx_{k+1}+s)^2}{R^2},$$

and

$$x_k - x_{k+2} = -\int_{v_{k+2}}^{v_k} (rx+s)^2 dv \asymp \frac{(rx_{k+1}+s)^2}{R^2},$$

where we have used (5.17) to estimate the range of integration. We use the interpolation mean value theorem in the form

$$\begin{aligned}&(x_{k-2} - x_k)\,(g(x_{k+1}) - \gamma) - (x_{k-2} - x_{k+1})\,(g(x_k) - \gamma) + \\ &+ (x_k - x_{k+1})\,(g(x_{k-2}) - \gamma) = \\ &= \frac{1}{2}(x_{k-2} - x_k)(x_{k-2} - x_{k+1})(x_k - x_{k+1})g''(\xi).\end{aligned}$$

for some ξ between x_{k+1} and x_{k-2}. Hence

$$g(x_{k+1}) - \gamma \ll \frac{rx_{k+1}+s}{H} + \frac{(rx_{k+1}+s)^4}{R^2}|g''(\xi)|.$$

Now ξ is an intermediate value between values x_i at which the Second Coincidence Condition holds. Lemma 4.1 says the the set on which the Second Coincidence Condition holds in its analytic form is an interval, and the bound (4.20) of Lemma 4.2 is valid at $x = \xi$, with $\Delta_2 \asymp R^2/NH$. We deduce that

$$g(x_{k+1}) - \gamma \ll \frac{rx_{k+1}+s}{H}. \tag{5.20}$$

Multiplication by t_i gives (5.19) with a larger constant. Since $q_i \ll H/B_4$ by the condition (5.6), the left hand side of (5.19) is numerically less than $1/2$. Hence

(5.18) implies (5.19) with the same implied constant, independent of i, and (5.20) holds uniformly in k. This completes the induction step.

For the second half of the proof we suppose that the constants in (5.12) to (5.25) are so large that Lemma 4.2 implies Lemma 5.2 for $L \leqslant 24$. For $L \geqslant 25$ we use a subsequence $i_o, \ldots, i_8$ of the Farey arcs $I(a_i/q_i)$, with $i_j = [j(L-1)/8]$, so that

$$i_{j+1} - i_j \geqslant \frac{L-1}{8} - 1 \geqslant \frac{2L}{25}.$$

For $j = 1, \ldots, 8$ let θ_j be the corresponding value of x_i. For x between θ_8 and θ_1 we have

$$LN \asymp \frac{NR^2}{r(rx+s)}, \qquad rx + s \asymp \frac{R^2}{Lr},$$

and

$$\theta_j - \theta_{j+1} = \int_{v_{i_j}}^{v_{i_{j+1}}} (rx+s)^2 dv \gg \frac{R^2}{Lr^2}.$$

For each $k = 1, \ldots, 4$ there is a ξ_k between θ_{2k} and θ_{2k+1} with

$$g'(\xi_k) = \frac{g(\theta_{2k-1}) - g(\theta_{2k})}{\theta_{2k-1} - \theta_{2k}} \ll \frac{R^2}{HLr} \cdot \frac{Lr^2}{R^2} \ll \frac{r}{H}. \tag{5.21}$$

The next derivative $g''(x)$ changes sign at most once, so the range for x can be divided into at most two subintervals on which $g'(x)$ is monotone. At least two consecutive ξ_k fall into the same subinterval. If for example ξ_1 and ξ_2 fall into the same subinterval, then $g'(x)$ is monotone between ξ_1 and ξ_2, and we have

$$g'(\theta_1) \ll \frac{r}{H}, \qquad g(\theta_1) - \gamma \ll \frac{r\theta_1 + s}{H} \ll \frac{R^2}{HLr}.$$

The value $x = \theta_1$ satisfies (5.16), and the corresponding point (y, z) satisfies (5.14) and (5.15) with $c = 0$, $d = h$.

Now we take $k = 2\ell$ and $k = 2\ell - 1$ in (5.21) for $\ell = 1, 2$. There is an η_ℓ between $\xi_{2\ell}$ and $\xi_{2\ell-1}$ with

$$g''(\eta_\ell) = \frac{g'(\xi_{2\ell-1}) - g'(\xi_{2\ell})}{\xi_{2\ell-1} - \xi_{2\ell}} \ll \frac{r}{H} \cdot \frac{Lr^2}{R^2} \ll \frac{Lr^3}{HR^2}. \tag{5.22}$$

Therefore there is a ζ between η_2 and η_1 with

$$g^{(3)}(\zeta) = \frac{g''(\eta_2) - g''(\eta_1)}{\eta_2 - \eta_1} \ll \frac{Lr^3}{HR^2} \cdot \frac{Lr^2}{R^2} \ll \frac{L^2 r^5}{HR^4}. \tag{5.23}$$

Now (5.22) is equivalent to

$$\begin{aligned} &\log h_1(x) - \log h_2(x) + 3\log(rx+s) - 3\log(r'x+s') \ll \\ &\ll \frac{(rx+s)^3}{NR^2} |g''(x)| \ll \frac{R^2}{HL^2N} \end{aligned} \tag{5.24}$$

for $x = \eta_1$ and η_2. From the analysis in Lemma 4.1, if the magic matrix is upper or lower triangular, then (5.24) holds for all x, which implies (5.12). If the magic matrix is type 3, then (5.24) holds for an interval of x which includes η_1 and η_2. Putting $x = \zeta$, we see from (5.23) and (5.24) that

$$g^{(3)}(\zeta) + \frac{r}{r\zeta + s} g''(\zeta) \ll \frac{L^2 r^5}{HR^4}.$$

By (4.24) and (4.26) in the proof of Lemma 4.2, we have

$$\frac{2C}{h_2(\zeta)(r\zeta + s)(r'\zeta + s')^4} \ll \frac{L^2 r^5}{HR^4} + \frac{L^2 r^5 N^2}{MR^6},$$

so

$$C \ll \frac{R^4}{HL^3 N} + \frac{NR^2}{M}.$$

By (4.12) we can drop the second term in the upper bound for C, so we have (5.13). If we take the constant in (5.12) large enough, then the interval on which the Second Coincidence Condition (4.3) holds to an accuracy (5.12) extends for at least L Farey arcs on each side of $x = \zeta$ by (4.13). We note that (5.12) is consistent with the requirement that $\Delta_2 \geqslant 1/M$ of (4.6) when (5.11) holds. ■

Lemma 5.3. (points close to long resonance curves) *Let* (5.5) *hold, and let* $C = C(e/r, f/s; e'/r', f'/s')$ *be a resonance curve. Suppose that there is a block of* L *consecutive Farey arcs* $I(a/q)$ *on which the Coincidence Conditions* (4.3) *to* (4.7) *hold, corresponding to an interval* J *of values of* x, *and for all* x *on* J

$$K(x) = \frac{1}{3h_1(x)r(rx+s)} \leqslant \sqrt{\frac{MNR^2}{H}}, \tag{5.25}$$

and the denominators q *lie in some range* $Q \leqslant q \leqslant 2Q$ *satisfying* (5.6). *Then there is an integer point* (c,d) *and an* x' *in* J *for which the point* (y', z') *on the resonance curve satisfies*

$$y' = c + O\left(\frac{R^2}{HL(rx'+s)}\right), \tag{5.26}$$

$$z' = d + O\left(\frac{R^2 x'}{HL(rx'+s)}\right). \tag{5.27}$$

Moreover either

$$\min_{x'/2 \leqslant x \leqslant 2x'} |g''(x)| \leqslant \frac{B_5 R^2}{HLx'(rx'+s)} \tag{5.28}$$

for some constant B_5, *sufficiently large in terms of the bounds for the derivatives of the underlying function* $F(x)$, *or there is a value* $x = x_1$ *at which* $y = c$ *and*

$$z = G(c) = d + O\left(\frac{R^2 x_1}{HL(rx_1+s)}\right) \tag{5.29}$$

and a value $x = x_2$ at which $z = d$ and

$$y = G^{-1}(d) = c + O\left(\frac{R^2}{HL(rx_2 + s)}\right), \tag{5.30}$$

where $z = G(y)$ is the equation of the resonance curve.

Proof. If $L \leqslant 100$, then (5.26) and (5.27) follow from Lemma 4.2. For $L > 100$ we number the Farey arcs $I(a_i/q_i)$. Let $\alpha = [L/4]$, $\beta = [3L/4]$, and let J' be the interval $[a_\alpha/q_\alpha, a_\beta/q_\beta]$. Let c/g be the rational number of least denominator in J'. Then c/g is a_i/q_i for some i. If $i \leqslant L/2$, then we take $e_0/r_0 = c/g$, and we take f_0/s_0 to be the successor of e_0/r_0 in the Farey sequence $\mathcal{F}(r_0)$. The resonance curve $C_0 = C(e_0/r_0, f_0/s_0; e_0'/r_0', f_0'/s_0')$ represents all the Farey arcs $I(a_j/q_j)$ with $i \leqslant j \leqslant \beta$, at least $(L+1)/4$ arcs. By Lemma 5.2 there is an integer point (c_0, d_0) close to C_0; for some x_0 the corresponding point (y_0, z_0) has

$$y_0 = c_0 + O\left(\frac{r_0}{H}\right), \qquad z_0 = d_0 + O\left(\frac{R^2}{HLr_0}\right), \tag{5.31}$$

If $j > L/2$, then we take $f_0/s_0 = c/g$, and e_0/r_0 to be the predecessor of f_0/s_0 in the Farey sequence $\mathcal{F}(s_0)$. We apply Lemma 5.2 to $C_0^* = C(f_0/s_0, e_0/r_0; f_0'/s_0', e_0'/r_0')$, and then we use Lemma 4.5 to deduce an integer point close to C_0 with

$$y_0 = c_0 + O\left(\frac{R^2}{HLs_0} + \frac{r_0}{NR^2}\left(1 + \frac{K_0^2}{M}\right)\right), \tag{5.32}$$

$$z_0 = d_0 + O\left(\frac{s_0}{H} + \frac{s_0}{NR^2}\left(1 + \frac{K_0^2}{M}\right)\right), \tag{5.33}$$

where we write

$$K_0 = \frac{1}{3h_3(0)r_0s_0} = \frac{1}{3h_3(0)}\left(\frac{f_0}{s_0} - \frac{e_0}{r_0}\right) \leqslant \frac{1}{3h_3(0)}\left(\frac{f_0}{s_0} - \frac{e}{r}\right) = K(x_1),$$

where x_1 is the value of x with $(ex + f)/(rx + s) = f_0/s_0$. Since e_0/r_0 is between e/r and f/s, the fraction f/s lies in the Farey sequence $\mathcal{F}(s_0)$, so $f_0/s_0 \leqslant f/s$, and $x_1 \geqslant 0$. There are at least $(L+1)/4$ Farey arcs represented by the resonance curve C_0, so by (5.25)

$$LN \ll K_0 \leqslant \sqrt{\frac{MNR^2}{H}}, \tag{5.34}$$

and

$$r_0s_0 \asymp \frac{NR^2}{K_0} \ll \frac{R^2}{L}; \tag{5.35}$$

the inequality (5.35) is also valid in the case $i \leqslant L/2$. Using (5.34) and (5.35) with (5.5), we can simplify (5.32) and (5.33) to

$$y_0 = c_0 + O\left(\frac{R^2}{HLs_0}\right), \qquad z_0 = d_0 + O\left(\frac{s_0}{H}\right). \tag{5.36}$$

In both cases, for x in J' we have

$$\frac{e}{r} + \frac{L-3}{4R^2} \leqslant \frac{ex+f}{rx+s} \leqslant \frac{f}{s} - \frac{L-3}{4R^2},$$

so that

$$\frac{1}{rx+s} \geqslant \frac{Lr}{5R^2}, \qquad \frac{x}{rx+s} \geqslant \frac{Ls}{5R^2}. \tag{5.37}$$

We have $e/r \leqslant e_0/r_0 < f_0/s_0 \leqslant f/s$, so we can write

$$\begin{pmatrix} f_0 & e_0 \\ s_0 & r_0 \end{pmatrix} = \begin{pmatrix} f & e \\ s & r \end{pmatrix} \begin{pmatrix} \ell & k \\ j & h \end{pmatrix}$$

as in Lemma 4.6 with $h,\ \ell \geqslant 1,\ j,k \geqslant 0,\ h\ell - jk = 1$. If $k > 0$ then we have

$$\frac{j}{h} < \frac{s_0}{r_0} < \frac{\ell}{k} < \frac{s_0}{r_0} + \frac{1}{hk} \leqslant \frac{2s_0}{r_0}, \tag{5.38}$$

so

$$\ell \leqslant \frac{2ks_0}{r_0} \ll \frac{kNR^2}{K_0 r_0^2} \ll \frac{kR^2}{Lr_0^2}. \tag{5.39}$$

Even if $k = 0$, we still have the first inequality in (5.38), so

$$j < \frac{hs_0}{r_0} < \frac{s_0}{r} \ll \frac{NR^2}{K_0 r r_0} \ll \frac{R^2}{Lrr_0}. \tag{5.40}$$

Lemma 4.6 lifts the point (c_0, d_0) close to C_0 to a point (c, d) close to some point (y', z') on C where the gradient of C is $-x'$. In the case (5.32) we have

$$y' = c + O\left(\frac{r}{NR^2}\left(1 + \frac{K_1^2}{M}\right)\right) + O\left(\frac{\ell r_0}{H} + \frac{kR^2}{HLr_0}\right), \tag{5.41}$$

$$z' = d + O\left(\frac{s}{NR^2} + \frac{K_1}{Mr}\right) + O\left(\frac{jr_0}{H} + \frac{hR^2}{HLr_0}\right), \tag{5.42}$$

where

$$K_1 = K\left(\frac{h}{k}\right) = \frac{k}{3\mu_0 r r_0}.$$

In the case (5.36) we have

$$y' = c + O\left(\frac{r}{NR^2}\left(1 + \frac{K_1^2}{M}\right)\right) + O\left(\frac{ks_0}{H} + \frac{\ell R^2}{HLs_0}\right), \tag{5.43}$$

$$z' = d + O\left(\frac{s}{NR^2} + \frac{K_1}{Mr}\right) + O\left(\frac{hs_0}{H} + \frac{jR^2}{HLs_0}\right). \tag{5.44}$$

Let $K' = K(x')$. Then by (5.25), (5.5), and (5.37)

$$\frac{r}{NR^2}\left(1+\frac{K_1^2}{M}\right) \leqslant \frac{r}{NR^2}\left(1+\frac{K'^2}{M}\right) \ll \frac{r}{NR^2}+\frac{r}{H} \ll \frac{r}{H} \ll \\ \ll \frac{R^2}{HL(rx'+s)},$$

and

$$\frac{s}{NR^2}+\frac{K_1}{Mr} \ll \frac{s}{NR^2}+\frac{K'}{Mr} \ll \frac{s}{H}+\frac{NR^2}{HK'r} \ll \frac{rx'+s}{H} \ll \\ \ll \frac{rx'}{H}\cdot\frac{R^2}{L(rx'+s)r}+\frac{s}{H}\cdot\frac{R^2x'}{Ls(rx'+s)} \ll \frac{R^2x'}{HL(rx'+s)}.$$

By (5.39) and (5.40), if $k \neq 0$, then

$$\frac{\ell r_0}{H}+\frac{kR^2}{HLr_0} \ll \frac{kR^2}{HLr_0} \ll \frac{K_1 r}{HLN} \ll \frac{K'r}{HLN} \ll \frac{R^2}{HL(rx'+s)},$$

and if $k = 0$, then $\ell = 1$, $r = r_0$ and

$$\frac{\ell r_0}{H}+\frac{kR^2}{HLr_0} = \frac{r}{H} \ll \frac{R^2}{HL(rx'+s)}.$$

Next we have

$$\frac{jr_0}{H}+\frac{hR^2}{HLr_0} \ll \frac{hs_0}{H}+\frac{hR^2}{HLr_0} \ll \frac{hR^2}{HLr_0}.$$

Now by (5.37)

$$\frac{h}{sr_0} = \frac{f}{s}-\frac{e_0}{r_0} \leqslant \frac{f}{s}-\frac{ex'+f}{rx'+s}+O\left(\frac{L}{R^2}\right) \ll \\ \ll \frac{f}{s}-\frac{ex'+f}{rx'+s} = \frac{x'}{s(rx'+s)}, \tag{5.45}$$

so we have

$$\frac{jr_0}{H}+\frac{hR^2}{HLr_0} \ll \frac{R^2x'}{HL(rx'+s)}.$$

We have as above if $k \neq 0$

$$\frac{ks_0}{H}+\frac{\ell R^2}{HLs_0} \ll \frac{kR^2}{HLr_0} \ll \frac{R^2}{HL(rx'+s)}.$$

Now by the construction of the interval J' in the case $i > L/2$,

$$\frac{1}{rs_0} = \frac{1}{r(rx'+s)}+O\left(\frac{L}{R^2}\right) \ll \frac{1}{r(rx'+s)},$$

so if $k = 0$, then $h = \ell = 1$ and

$$\frac{ks_0}{H} + \frac{\ell R^2}{HLs_0} = \frac{R^2}{HLs_0} \ll \frac{R^2}{HL(rx' + s)}.$$

Also in the case $i > L/2$

$$\frac{j}{ss_0} = \frac{f}{s} - \frac{f_0}{s_0} < \frac{f}{s} - \frac{ex' + f}{rx' + s} = \frac{x'}{s(rx' + s)},$$

so

$$\frac{jR^2}{HLs_0} \ll \frac{R^2 x'}{HL(rx' + s)},$$

and by (5.35) and (5.45)

$$\frac{hs_0}{H} \ll \frac{hR^2}{HLr_0} \ll \frac{R^2 x'}{HL(rx' + s)}.$$

Hence the error terms in (5.41), (5.42), (5.43), and (5.44) can all be estimated as in (5.26) and (5.27).

Suppose that (5.28) is false. Then since

$$y - y' = -\int_{x'}^{x} g''(x)dx,$$

the values of y' for $x'/2 \leqslant x \leqslant 2x'$ include an interval $[y' - \delta, y' + \delta]$ with

$$\delta = \frac{B_5 R^2}{2HL(rx' + s)}.$$

If B_5 is so large that δ is greater than the error term in (5.26), then $y = c$ for some value $x = x_1$ with $x'/2 \leqslant x_1 \leqslant 2x'$. The corresponding value $z = z_1$ has

$$\frac{z_1 - z'}{c - y'} = -\xi$$

for some ξ between z_1 and z'. We deduce (5.29). A similar argument shows that $z = d$ for some value $x = x_2$, and (5.30) holds. ∎

Our next two lemmas address points that were overlooked in [3]. We count coincidences by estimating the number of integer points close to resonance curves. In [2] and [3] it seemed obvious that an integer point close to the curve corresponds to at most one interval of coincident Farey arcs. In Lemma 5.4 we prove a little less, that there are at most seven such intervals. Lemma 5.5, that any two integer points close to the resonance curve differ in both coordinates c and d, is implicit in the calculations of [3]. The statement and proof of Lemma 5.5 were omitted in error from the detailed account in [3].

Lemma 5.4. (coincidences with the same integer point) *In Lemma* 4.3 *the Farey arcs for which the four Coincidence Conditions hold, and the integers* c, d *take fixed values, fall into at most seven disjoint intervals, such that if* $I(a/q)$ *is any Farey arc in one of those intervals, then the four Coincidence Conditions, weakened by a bounded factor, hold on* $I(a/q)$.

Proof. We divide the resonance curve into at most seven regions. By Lemma 5.1 there is at most one point x_0 with $g''(x_0) = 0$. The corresponding point (y_0, z_0) on the resonance curve is a cusp. The curve has one concave and one convex branch. We consider each branch separately. Each branch has negative gradient, so the lines $y = c$, $z = d$ divide it into at most three parts. Finally, if the points where $x = s/2r$ and $x = 2s/r$ occur in the same region, then we divide the region at the point where $x = s/r$. On each part of the curve the numbers $g''(x)$, $y - c$, and $z - d$ have constant sign, and either $x \leqslant 2s/r$ or $x \geqslant s/2r$ holds throughout.

Now suppose that there are two Farey arcs $I(a_1/q_1)$ and $I(a_2/q_2)$ at which the four Coincidence Conditions hold, with x_1 and x_2 in the same region of the curve, corresponding to the same integer point (c, d) in Lemma 4.3. Consider a general value $x = u/t$ in $x_1 \leqslant x \leqslant x_2$. In the notation of Lemma 3.4 and Lemma 4.3 we have

$$L - L' = \beta t + \alpha u - tg\left(\frac{u}{t}\right) + O\left(\frac{q}{NR^2}\left(1 + \frac{K^2(x)}{M}\right)\right) =$$
$$= \beta t + \alpha u - tg\left(\frac{u}{t}\right) + O\left(\frac{q}{H}\right),$$

by the assumptions (4.32) and (4.28). At $x = x_1$ and x_2 we actually have

$$L - L' = dt + cu + O\left(\frac{q}{H}\right). \tag{5.46}$$

Subtracting and dividing by t, we have for $x = x_1$ and x_2

$$\beta - d + (\alpha - c)x - g(x) \ll \frac{ru + st}{Ht} \ll \frac{rx + s}{H}. \tag{5.47}$$

The derivative of the left hand side of (5.46) is $\alpha - c - g'(x) = y - c$, which has constant sign, so the left hand side is monotone for $x_1 \leqslant x \leqslant x_2$. For $x \leqslant 2s/r$ we deduce that (5.46) holds for $x_1 \leqslant x \leqslant x_2$. In the case $x \geqslant s/2r$ we put $X = t/u = 1/x$, and we use

$$(\beta - d)x + \alpha - c - Xg\left(\frac{1}{X}\right) \ll \frac{q}{Hu} \ll \frac{r + sX}{H} \ll \frac{r}{H}. \tag{5.48}$$

The derivative with respect to X of the left hand side of (5.48) is

$$\beta - d - g\left(\frac{1}{X}\right) + \frac{1}{X}g'\left(\frac{1}{X}\right) = z - d,$$

so the left hand side is monotone, and (5.48) holds for $x_1 \leqslant x \leqslant x_2$. In both cases we have (5.46), the Fourth Coincidence Condition weakened by a bounded factor.

For the Third Coincidence Condition, by Lemma 3.4

$$\begin{aligned} &\left(\!\!\left(\frac{\bar{a}b_2}{q} - \frac{\overline{a'}b'_2}{q'}\right)\!\!\right) = \\ &= \left(\!\!\left(-\frac{1}{t}g'\left(\frac{u}{t}\right) - \frac{\bar{u}(\ell-\ell')}{t} + \frac{\alpha}{t} - \frac{\kappa_2 r}{t(ru+st)} + \frac{\kappa'_2 r'}{t(r'u+s't)}\right)\!\!\right) + \\ &+ O\left(\frac{1}{q} + \frac{1}{K(x)q}\right). \end{aligned} \tag{5.49}$$

The assumption (4.27) of Lemma 4.3 implies (4.35), so $K(x) \gg N$, and we may drop the second error term in (5.49). For $x_1 \leqslant x \leqslant x_2$ we have (5.46), so (4.38) gives

$$\begin{aligned} &\frac{\kappa'_2 r'}{t(r'u+s't)} - \frac{\kappa_2 r}{t(ru+st)} = \frac{(\kappa'_2-\kappa_2)r}{t(ru+st)} - \frac{\kappa'_2 C}{(ru+st)(r'u+s't)} \ll \\ &\ll \frac{q}{H}\cdot\frac{r}{tq} + \frac{R^4}{HNq^2} \ll \frac{1}{H}\cdot\frac{NR^2}{K(x)q} + \frac{R^2}{Hq} \ll \frac{R^2}{Hq}, \end{aligned}$$

where we have used (4.35) again. Next we note that

$$\frac{c}{t} - \frac{\bar{u}(\ell-\ell')}{t} = \frac{c-\bar{u}(cu+dt)}{t} = c\bar{t} - d\bar{u},$$

an integer, so (5.49) simplifies to

$$\left(\!\!\left(\frac{\bar{a}b_2}{q} - \frac{\overline{a'}b'_2}{q'}\right)\!\!\right) = \left(\!\!\left(\frac{\alpha}{t} - \frac{c}{t} - \frac{1}{t}g'\left(\frac{u}{t}\right)\right)\!\!\right) + O\left(\frac{R^2}{Hq}\right).$$

For $x = x_1$ or x_2 the Third Coincidence Condition holds, so (4.33) asserts that

$$y - c = \alpha - c - g'(x) \ll \frac{R^2}{H(rx+s)} \tag{5.50}$$

at $x = x_1$ and x_2. The derivative of the left hand side of (5.50) is $-g''(x)$, which has constant sign, so the left hand side is monotone for $x_1 \leqslant x \leqslant x_2$. For $x \leqslant 2s/r$ we deduce (5.50) for $x_1 \leqslant x \leqslant x_2$. The other case is $s/2r \leqslant x \ll R^2/r^2$; the upper bound for x comes from (4.27). The combination (5.47) minus x times (5.50) gives

$$\beta - d + xg'(x) - g(x) \ll \frac{rx}{H} + \frac{R^2}{Hr} \ll \frac{R^2}{Hr} \tag{5.51}$$

at $x = x_1$ and x_2. The derivative of the left hand side of (5.51) is $xg''(x)$, which has constant sign, so the left hand side is monotone and (5.51) holds for $x_1 \leqslant x \leqslant x_2$. Subtracting from (5.47) and dividing by x, we recover (5.50) for $x_1 \leqslant x \leqslant x_2$. The steps from (5.49) to (5.50) are reversible, so we have

$$\left(\!\!\left(\frac{\bar{a}b_2}{q} - \frac{\overline{a'}b'_2}{q'}\right)\!\!\right) \ll \frac{R^2}{HQ}$$

for $x_1 \leqslant x \leqslant x_2$, the Third Coincidence Condition weakened by a bounded factor.

■

Lemma 5.5. (uniqueness of coordinates) *Let L and Q be positive integers, with Q satisfying* (5.6). *Let B_6 be a constant sufficiently large in terms of the derivatives of the underlying function $F(x)$. Then on the part of the resonance curve with*

$$K(x) \leqslant \frac{HLN}{B_6 Q}, \tag{5.52}$$

among the integer points (c, d) corresponding to blocks of at least L consecutive Farey arcs $I(a/q)$, each with $q \leqslant 2Q$, strictly between the Farey arcs $I(e/r)$ and $I(f/s)$, the integer c determines the integer d uniquely, and the integer d determines the integer c uniquely.

Proof. If x corresponds to a Farey arc $I(a/q)$ in such a block, then

$$\frac{1}{2Qr} \leqslant \frac{a}{q} - \frac{e}{r} = \frac{ex+f}{rx+s} - \frac{e}{r} = \frac{1}{r(rx+s)},$$
$$\frac{1}{2Qr} \leqslant \frac{f}{s} - \frac{a}{q} = \frac{f}{s} - \frac{ex+f}{rx+s} = \frac{x}{r(rx+s)},$$

so $rx + s \leqslant \min(2Q, 2Qx)$, and since $r \leqslant Q$, we have

$$\frac{s}{Q} \leqslant x \leqslant \frac{2Q}{r}. \tag{5.53}$$

If $I(a_1/q_1)$, $I(a_2/q_2)$ (with $x_1 < x_2$) lie in blocks with the same c but with different d_1 and d_2, then in (5.27)

$$d_i - z_i \ll \frac{R^2 x_i}{HL(rx_i + s)} \ll \frac{R^2}{HLr} \ll \frac{1}{B_4},$$

so for B_4 sufficiently large we have $|d_i - z_i| \leqslant 1/4$. Hence for some ξ in $x_1 < \xi <$ $< x_2$ we have

$$\frac{1}{2} \leqslant |z_1 - z_2| = |\xi(y_1 - y_2)| \leqslant x_2|y_1 - y_2| \leqslant$$
$$\leqslant \frac{2Q}{r}|y_1 - y_2| \ll \frac{Q}{r}.\frac{R^2}{HL(rx_1 + s)} \ll \frac{Q}{HLN}K(x_1) \ll \frac{1}{B_6},$$

which is impossible if B_6 is sufficiently large.

Similarly if $I(a_1/q_1)$, $I(a_2/q_2)$ (with $x_1 < x_2$) lie in blocks with the same d but with different c_1 and c_2, then in (5.26)

$$c_i - y_i \ll \frac{R^2}{HL(rx_i + s)} \ll \frac{r}{HLN}K(x_i) \ll \frac{r}{B_6 Q} \ll \frac{1}{B_6},$$

and since B_6 is sufficiently large, we have $|c_i - y_i| \leqslant 1/4$. Thus for some ξ in $x_1 < \xi < x_2$ we have

$$\frac{1}{2} \leqslant |y_1 - y_2| = \frac{1}{\xi}|z_1 - z_2| \ll \frac{1}{\xi}.\frac{R^2}{HL}.\frac{x_2}{rx_2 + s} \ll \frac{R^2}{HLr\xi},$$

and we have

$$x_1 \leqslant \xi \ll \frac{R^2}{HLr}. \tag{5.54}$$

There are now two cases. If $rx_1 \geqslant s$, then (5.52) with $x = x_1$ gives

$$\frac{NR^2}{r^2 x_1} \ll \frac{HLN}{B_6 Q},$$

and so

$$\frac{R^2}{r^2} \ll \frac{HLx_1}{B_6 Q} \ll \frac{R^2}{B_6 Qr},$$

which is impossible if B_6 is sufficiently large. If $rx_1 \leqslant s$, then (5.52) with $x = x_1$ gives

$$\frac{NR^2}{rs} \ll \frac{HLN}{B_6 Q},$$

and (5.53) and (5.54) lead to the similar contradiction

$$\frac{B_6 R^2}{HLr} \ll \frac{s}{Q} \leqslant x_1 \ll \frac{R^2}{HLr}.$$

■

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Address: School of Mathematics, University of Cardiff, 23 Senghennydd Road, Cardiff CF24 4YH, Wales, Great Britain

E-mail: huxley@cf.ac.uk

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