Functiones et Approximatio
XXXII (2004), 79–98

THE WALSH TRANSFORM OF WAVELET TYPE SYSTEMS: CONVERGENCE ALMOST EVERYWHERE

Barbara Wolnik

Abstract: The main results of the paper are the following: the Fourier expansion of $f \in L_p$, $1 < p < \infty$, with respect to the Walsh transform of a wavelet type system converges a.e. to f and if $f \in L_1$ then the same is true for the Cesàro means.
Keywords: Uniformly bounded systems, convergence almost everywhere, Cesàro summability.

1. Introduction

The subject of this paper is to study pointwise convergence of Fourier expansions with respect to the Walsh transform of a wavelet type system on $[0,1]$ or $\mathbb{T}$. By a wavelet type system we mean a biorthogonal system of functions which have dyadic scaled estimates. The Walsh transform of a wavelet type system is the system which arises from a wavelet type system in the same way as the Walsh system arises from the Haar system. It appears that this new system is uniformly bounded.

This method has been first used by Z. Ciesielski [4] to construct a bounded system of polygonals starting from the Franklin system. An analogous construction has been applied by S. Ropela [17] to orthogonal spline bases. He has obtained bounded orthogonal spline systems (called Ciesielski's systems) and has proved that these systems are bases in L_p for $1 < p < \infty$. In [5] Z. Ciesielski has used this construction to the biorthogonal splines. The problem of pointwise convergence of Fourier expansions with respect to the Walsh system has been considered by P. Billard in [2] ($p = 2$) and P. Sjölin in [19] ($p > 1$), with respect to the Walsh transform of the Franklin system by Z. Ciesielski, P. Simon and P. Sjölin in [8] and in the Walsh transform of arbitrary spline system by Z. Ciesielski in [5]. We extend their results to the Walsh transform of wavelet type systems in Theorem 3.4, which states that the expansion of $f \in L_p$ for $1 < p < \infty$ converges a.e. to f.

2001 Mathematics Subject Classification: primary 42C15, 40A30, 40G05, secondary 41A15, 42C40.

The situation is different when $f \in L_1(0,1)$ is considered. By the well-known result of S.V. Bočkariev [3] for each uniformly bounded ONS $\{f_n\}_{n\in\mathbb{N}}$ on $[0,1]$ there is a function $f \in L_1(0,1)$ whose Fourier series in the system $\{f_n\}_{n\in\mathbb{N}}$ diverges unboundedly at every point of a set $E \subset [0,1]$ of positive measure. Moreover, K.S. Kazarian and A.S. Sargsian [13] proved that there exists a function from $L_1(0,1)$, whose Fourier series in the bounded system of polygonals introduced by Z. Ciesielski diverges a.e. on $[0,1]$. This result was extended to the Walsh transform of biorthogonal wavelet type systems by A. Kamont and the author in [12].

Therefore, in the case of functions from L_1 various methods of summation are studied. In this direction, N.J. Fine [10] has proved that the Cesàro means of the Fourier series of any $f \in L_1(0,1)$ with respect to the Walsh system converge to f a.e. on $[0,1]$ and F. Weisz [20] has proved this fact in the case of the Walsh transform of spline systems. It occures that this result can be extended to the Walsh transform of arbitrary wavelet type systems (see Theorem 4.1).

2. Preliminaries and notation

Let $(\mathbb{I}, d)$ denote either the metric space $([0,1], d_1)$ or $(\mathbb{T}, d_2)$, where

$$d_1(x,y) = |x-y|, \quad x, y \in [0,1], \quad d_2(x,y) = \min(|x-y|, 1-|x-y|), \quad x, y \in \mathbb{T}.$$

By $I_{j,k}$ we will denote the interval $[\frac{k-1}{2^j}, \frac{k}{2^j}]$ and for $n \in \mathbb{N}$ we define $n * I_{j,k}$ as the set $\{x \in \mathbb{I} : \ d(x, \frac{k}{2^j}) \leqslant \frac{n}{2^j}\}$.
By a biorthogonal wavelet type system on $\mathbb{I}$ we mean a biorthogonal system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$, where natural $N \geqslant -1$ is given, consisting of functions on $\mathbb{I}$ satisfing the following conditions:

(I) There is a constant $M > 0$ such that for any $n \in \{-N, \dots, 0, 1\}$ and $x \in \mathbb{I}$

$$|\psi_n(x)| \leqslant M \quad \text{and} \quad |\phi_n(x)| \leqslant M.$$

(II) For $j \geqslant 0$, $k \in \{1, 2, \dots, 2^j\}$ and $x \in \mathbb{I}$

$$|\psi_{2^j+k}(x)| \leqslant 2^{\frac{j}{2}} S(2^j d(x, \frac{k}{2^j})), \quad |\phi_{2^j+k}(x)| \leqslant 2^{\frac{j}{2}} S(2^j d(x, \frac{k}{2^j})),$$

where

(III) $S : [0, \infty) \to \mathbb{R}$ is a nonincreasing function such that

$$\int_0^{\infty} \ln(1+x) S(x) dx < +\infty.$$

In this paper we will suppose an additional condition, namely, that the system $\{\psi\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$.

As we will see, conditions (I), (II), (III) and the fact that the system $\{\psi\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$ are sufficient to secure a lot of good properties of wavelet type system. Let us note that many classical or newly introduced systems satisfy these conditions. The most known ones are: the Haar system, the Franklin system, the orthogonal and biorthogonal bases discussed in [6], adaptations of Daubechies' wavelets to the interval [0, 1] (see [1], [9], [15]), periodic wavelets (see for instance [21]) and the Petrushev systems [16] consisting of rational functions of uniformly bounded degrees.

Remark. As a consequence of the monotonicity of S and of condition (III) one can get

Lemma 2.1. *There is a constant* C *such that*

$$\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^j S(k2^j) < C \tag{1}$$

and for $j \geqslant 0$ *and* $x, y \in \mathbb{I}$

$$\sum_{k=1}^{2^j} S(2^j d(x, \frac{k}{2^j}))S(2^j d(y, \frac{k}{2^j})) \leqslant CS(2^{j-1}d(x,y)). \tag{2}$$

Moreover,

$$\lim_{x\to\infty} \log(x+1)S(x) = 0. \tag{3}$$

Let $\{\chi_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ denote the Haar and Walsh functions, respectively. For any $j \geqslant 0$ we define the matrix (see for instance [18])

$$A_{k,l}^{(j)} = (w_{2^j+k}, \chi_{2^j+l}) = 2^{-\frac{j}{2}} w_k\left(\frac{2l-1}{2^{j+1}}\right), \quad k, l = 1, 2, \ldots, 2^j \tag{4}$$

which is orthogonal and symetric (the last fact was proved in [4]).
The Walsh transform of the wavelet type system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ is the system $\{\overset{b}{\psi}_n, \overset{b}{\phi}_n\}_{n=-N}^{\infty}$ given by formulae:
for $n \in \{-N, \ldots, 0, 1\}$

$$\overset{b}{\psi}_n(x) = \psi_n(x), \quad \overset{b}{\phi}_n(x) = \phi_n(x),$$

for $j \geqslant 0,\ k = 1, \ldots, 2^j$

$$\psi^b_{2^j+k}(x) = \sum_{l=1}^{2^j} A_{k,l}^{(j)} \psi_{2^j+l}(x), \quad \phi^b_{2^j+k}(x) = \sum_{l=1}^{2^j} A_{k,l}^{(j)} \phi_{2^j+l}(x).$$

Let $P_n : L_1(\mathbb{I}) \to L_1(\mathbb{I})$ denote the projections

$$P_n f = \sum_{i=-N}^{n} (f, \phi_i)\psi_i, \quad n \geqslant -N.$$

Theorem 2.2 below summarises necessary facts concerning the systems $\{\psi_n\}_{n=-N}^{\infty}$ and $\{\psi_n^b\}_{n=-N}^{\infty}$.

Theorem 2.2. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a biorthogonal system fulfilling conditions* (I), (II), (III). *In addition, assume that $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$. Then we have:*

(o) *The system $\{\psi_n\}_{n=-N}^{\infty}$ is an unconditional basis in $L_p(\mathbb{I})$ for $1 < p < \infty$. Moreover, the systems $\{\psi_n\}_{n=1}^{\infty}$ and $\{\chi_n\}_{n=1}^{\infty}$ are L_p-equivalent.*

(i) *The maximal operator*

$$P^* f(x) = \sup_{n \geqslant -N} |P_n f(x)|$$

is of type (p,p) for $1 < p < \infty$ and of the weak type $(1,1)$.

(ii) *For any $f \in L_1(\mathbb{I})$ we have $P_n f(x) \to f(x)$ a.e. on $\mathbb{I}$.*

(iii) *The system $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$ is a basis in $L_p(\mathbb{I})$ for $1 < p < \infty$.*

(iv) *The series*

$$\sum_{n=2}^{\infty} a_n \overset{b}{\psi}_n \quad \text{and} \quad \sum_{n=2}^{\infty} a_n w_n$$

are equiconvergent in $L_p(\mathbb{I})$ for $1 < p < \infty$ and their norms are equivalent.

The above properties of the wavelet type systems have been proved by the author in [22]. We have decided not to present here the proof of Theorem 2.2, since the methods of proofs are similar to the proofs of the analogous results in the case of wavelets on $\mathbb{R}$. More precisely, the unconditionality of $\{\psi_n\}_{n=-N}^{\infty}$ in $L_p(\mathbb{I})$ is proved analogously as the unconditionality of wavelet bases on $\mathbb{R}$ in P. Wojtaszczyk [21], and its L_p-equivalence to the Haar system is checked as in G.G. Gevorkyan, B. Wolnik [11]. The proofs of properties (i) and (ii) follow by arguments analogous to those used in S.E. Kelly, M.A. Kon and L.A. Raphael [14] in case of wavelets on $\mathbb{R}^n$. Once we know (o), properties (iii) and (iv) are obtained similarly as the corresponding results for the spline systems or the Franklin system in S. Ropela [17] and Z. Ciesielski and S. Kwapień [7].

3. Convergence a.e. for $f \in L_p(\mathbb{I})$, $1 < p < \infty$

We start with the theorem concerning the type (p,p) of maximal operator for the partial sums with respect to the system $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$. The crucial role in the proof is played by P. Sjölin's result [19] concerning (p,p)-type for the maximal operator for partial sums with respect to the Walsh system:

Theorem 3.1. (P. Sjölin [19]) *Let $1 < p < \infty$. There is a constant C_p such that*

$$\| \sup_n | \sum_{i=1}^{n} a_i w_i | \|_p \leqslant C_p \| \sum_{i=1}^{\infty} a_i w_i \|_p.$$

The type (p,p) for the maximal operator for the bounded orthonormal set of polygonals was proved in [8] and for the remaining Ciesielski's systems in [5]. We extend this result to the Walsh transform of arbitrary wavelet type systems.

Theorem 3.2. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a biorthogonal system fulfilling conditions* (I), (II), (III). *If $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$ then the maximal operator T^**

$$T^* f(t) := \sup_{n\geqslant -N} |\sum_{l=1}^{n} (f, \overset{b}{\phi_l}) \overset{b}{\psi_l}(t)| \tag{5}$$

is of type (p,p) for $1 < p < \infty$.

Before we begin the proof of Theorem 3.2 we show the following auxiliary result, which for the Franklin system was proved in [8]:

Lemma 3.3. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a biorthogonal system fulfilling conditions* (I), (II), (III). *Let r_j be the j-th Rademacher function. For $j \geqslant 0$ we define the functions on $\mathbb{I}^2$ by formulae*

$$G_j(t,s) = 2^{\frac{j}{2}} r_{j+1}(s) \psi_{2^j+k}(t) \quad \text{for} \quad \frac{k-1}{2^j} \leqslant s < \frac{k}{2^j} \quad \text{and} \quad k = 1, \ldots, 2^j. \tag{6}$$

Then there is a constant C such that for $h \in L_1(\mathbb{I})$ we have

$$|\int_{\mathbb{I}} G_j(t,s) h(s) ds| \leqslant C \mathcal{M} h(t), \quad t \in \mathbb{I},$$

where $\mathcal{M}h$ denotes the Hardy-Littlewood maximal function of h.

Proof. It follows directly from the definition of G_j that if $s \in I_{j,k}$, then

$$|G_j(t,s)| \leqslant 2^j S(2^j d(t, \frac{k}{2^j})) \leqslant \begin{cases} 2^j S(0) & \text{for } d(s,t) \leqslant \frac{2}{2^j} \\ 2^j S(2^{j-1} d(s,t)) & \text{for } d(s,t) > \frac{2}{2^j} \end{cases} \tag{7}$$

Now, let $t \in \mathbb{I}$. Then

$$|\int_{\mathbb{I}} G_j(t,s) h(s) ds| \leqslant \int_{d(s,t) \leqslant \frac{2}{2^j}} |G_j(t,s)||h(s)| ds + \int_{d(s,t) > \frac{2}{2^j}} |G_j(t,s)||h(s)| ds =:$$
$$=: B_1 + B_2.$$

For the first term we obtain from (7)

$$B_1 \leqslant S(0) 2^j \int_{d(s,t) \leqslant \frac{2}{2^j}} |h(s)| ds \leqslant C \mathcal{M} h(t).$$

For the second term we get using (7) and Lemma 2.1

$$\begin{aligned} B_2 &\leqslant \sum_{l=1}^{j-1} \int_{\frac{2^l}{2^j} \leqslant d(s,t) \leqslant \frac{2^{l+1}}{2^j}} 2^j S(2^{j-1} \frac{2^l}{2^j}) |h(s)| ds \leqslant \\ &\leqslant \sum_{l=1}^{j-1} S(2^{l-1}) 2^j \int_{d(s,t) \leqslant \frac{2^{l+1}}{2^j}} |h(s)| ds \leqslant \\ &\leqslant \sum_{l=1}^{j-1} S(2^{l-1}) 2^j \left| \left\{ s: \ d(s,t) \leqslant \frac{2^{l+1}}{2^j} \right\} \right| \mathcal{M} h(t) \leqslant \\ &\leqslant C \sum_{l=1}^{j-1} S(2^{l-1}) 2^{l+1} \mathcal{M} h(t) \leqslant C' \mathcal{M} h(t). \end{aligned}$$

■

The proof of Theorem 3.2: Introduce

$$\tilde{T}^* f(t) := \sup_{j\geqslant 0} \sup_{1\leqslant k\leqslant 2^j} |\sum_{l=1}^{k} (f, \overset{b}{\phi}_{2^j+l}) \overset{b}{\psi}_{2^j+l}(t)|.$$

Since for each $j \geqslant 0$ and $t, s \in \mathbb{I}$

$$\sum_{n=1}^{2^j} \phi_n^b(s)\psi_n^b(t) = \sum_{n=1}^{2^j} \phi_n(s)\psi_n(t)$$

then it follows from (i) of Theorem 2.2 that it is enough to show the type (p,p), $1 < p < \infty$, for the operator $\tilde{T}^*$.

Since $\overset{b}{\psi}_{2^j+k}(t) = \int_0^1 w_{2^j+k} G_j(t,s)ds$, thus from Lemma 3.3 we have

$$|\sum_{l=1}^{k} (f, \overset{b}{\phi}_{2^j+l}) \psi_{2^j+l}(t)| \leqslant C\mathcal{M}\Big(\sum_{l=1}^{k} (f, \overset{b}{\phi}_{2^j+l}) w_{2^j+l}\Big)(t)$$

which gives

$$\tilde{T}^* f(t) \leqslant C\mathcal{M} w^*(t), \tag{8}$$

where

$$w^*(t) = \sup_{j\geqslant 0} \sup_{1\leqslant k\leqslant 2^j} |\sum_{l=1}^{k} (f, \overset{b}{\phi}_{2^j+l}) w_{2^j+l}(t)|.$$

Using Theorem 3.1, (iv) of Theorem 2.2 and the fact that $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$ is a basis in $L_p(\mathbb{I})$ we get

$$\|w^*\|_p \leqslant 2C_p \| \sum_{i=1}^{\infty} (f, \overset{b}{\phi}_i) w_i \|_p \leqslant C \| \sum_{i=1}^{\infty} (f, \overset{b}{\phi}_i) \overset{b}{\psi}_i \|_p \leqslant C \|f\|_p. \tag{9}$$

Now, from (8), the (p,p)-type for the Hardy-Littlewood maximal function and (9) it follows that

$$\|\tilde{T}^* f\|_p \leqslant C \|\mathcal{M} w^*\|_p \leqslant C' \|w^*\|_p \leqslant C'' \|f\|_p, \tag{10}$$

which proves the type (p,p) of operator $\tilde{T}^*$. ■

Combining Theorem 3.2 with the usual density argument (see [18], Theorem 3.1.2) we get

Theorem 3.4. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a biorthogonal system fulfilling conditions* (I), (II), (III). *If $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$ then for $f \in L_p(\mathbb{I})$ $(1 < p < \infty)$ the series*

$$\sum_{n=-N}^{\infty} (f, \overset{b}{\phi}_n) \overset{b}{\psi}_n$$

converges a.e. to f.

4. Cesàro summability for $f \in L_1(\mathbb{I})$

In this part of the paper we will prove that the sequence of the arithmetic means of the Fourier series of f with respect to the system $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$ converges to f a.e. on $\mathbb{I}$.

Without loss of generality we can assume that $d = d_2$. (If the system $\{\psi_n\}$ fulfils condition (II) with d_1 then of course it fulfils (II) with d_2).

First we introduce a notation. The Walsh-Dirichlet and the Walsh-Fejér kernels are denoted by $\tilde{D}_n$ and $\tilde{F}_n$ respectively, i.e.

$$\tilde{D}_n(t,x) = \sum_{i=1}^{n} w_i(t)w_i(x), \quad \tilde{F}_n(t,x) = \frac{1}{n}\sum_{j=1}^{n} \tilde{D}_j(t,x).$$

It is known (see [18]), that if we denote by $\dot{+}$ the dyadic addition then

$$\tilde{D}_n(t,x) = D_n(t\dot{+}x) = \sum_{i=1}^{n} w_i(t\dot{+}x), \quad \tilde{F}_n(t,x) = F_n(t\dot{+}x) = \frac{1}{n}\sum_{j=1}^{n} D_j(t\dot{+}u)$$

and

$$D_{2^n}(x) = 2^n \mathbb{1}_{[0,2^{-n})}(x), \tag{11}$$

$$F_{2^n}(x) \leqslant \sum_{j=0}^{n} 2^{j-n} D_{2^n}(x\dot{+}2^{-j-1}). \tag{12}$$

Moreover, for $2^{N-1} \leqslant n < 2^N$

$$|F_n(x)| \leqslant 3\sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} D_{2^i}(x\dot{+}2^{-j-1}). \tag{13}$$

For the partial sums $S_n f$ and the Fejér means $\sigma_n f$ of the function f with respect to the system $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$ we have

$$S_n f(x) := \sum_{i=-N}^{n} (f, \overset{b}{\phi}_i)\overset{b}{\psi}_i(x) = \int_{\mathbb{I}} D_n^{\psi}(x,t)f(t)dt,$$

$$\sigma_n f(x) := \frac{1}{n}\sum_{j=1}^{n} S_j f(x) = \int_{\mathbb{I}} F_n^{\psi}(x,t)f(t)dt,$$

where

$$D_n^{\psi}(x,t) = \sum_{i=-N}^{n} \overset{b}{\phi}_i(t)\overset{b}{\psi}_i(x), \quad F_n^{\psi}(x,t) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=-N}^{j} \overset{b}{\phi}_i(t)\overset{b}{\psi}_i(x)$$

are the Dirichlet and Fejér kernels for the system $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$, respectively. (For the simplification in the definition of σ_n we consider the sum from $j = 1$, the partial sums $S_{-N}, ..., S_0$ can be ignored.)

The main result of this section is

Theorem 4.1. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a biorthogonal system fulfilling conditions* (I), (II), (III). *If $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$, then for any $f \in L_1(0,1)$ we have*

$$\sigma_n f \longrightarrow f \quad a.e.$$

Our method is based on the proof of Weisz's result [20] concerning the bounded Ciesielski systems, but there are some differences. F. Weisz proved that the maximal operator for Cesàro means is bounded from the Hardy space H_p to L_p for $1/2 < p < \infty$ and by the interpolation he obtained the weak-type $(1,1)$ of this operator. The usual density argument (see [18], Theorem 3.1.2) then implied the convergence result. His proof depends heavily on the estimates for the derivatives of basic functions. In our case we do not have such estimates. However, it occures that in case of wavelet type bases, it is possible to prove that the limes superior operator for the Cesàro means of the Fourier series with respect to the system $\{\overset{b}{\psi}_n\}_{n=-N}^{\infty}$ is of the weak-type $(1,1)$ (as a consequence of its quasi-locality). Moreover, this result is also sufficient to obtain the required convergence as it follows from the following weaker version of Theorem 3.1.2 from [18]:

Lemma 4.2. *Let X_0 be a dense subset of $L_1(\mathbb{I})$. Let T_n $(n \in \mathbb{N})$, S be linear operators from $L_1(\mathbb{I})$ to $L_0(\mathbb{I})$. Let us assume that the operator S is of the weak-type $(1,1)$ and that for any function $f \in X_0$ we have $\lim_{n\to\infty} T_n f = Sf$ a.e. on $\mathbb{I}$. If the operator T defined as*

$$Tf(x) := \limsup_{n\to\infty} |T_n f(x)|,$$

is also of the weak-type $(1,1)$, then for every function $f \in L_1(\mathbb{I})$ we have

$$\lim_{n\to\infty} T_n f = Sf \quad \text{a.e. on } \mathbb{I}.$$

Proof. Let us fix $f \in L_1(\mathbb{I})$. Let us choose $f_m \in X_0$ such that $\lim_{m\to\infty} \|f - f_m\|_1 = 0$. Since $\limsup_{n\to\infty} |T_n f_m - Sf_m| = 0$ a.e. on $\mathbb{I}$ hence

$$\limsup_{n\to\infty} |T_n f - Sf| \leqslant \limsup_{n\to\infty} |T_n(f - f_m)| + \limsup_{n\to\infty} |T_n f_m - Sf_m| + |Sf_m - Sf| = T(f - f_m) + |Sf_m - Sf|.$$

As T and S are of the weak-type $(1,1)$ we have

$$|\{x \in \mathbb{I} : \limsup_{n\to\infty} |T_n f(x) - Sf(x)| > 2y\}| \leqslant |\{x \in \mathbb{I} : T(f - f_m)(x) > y\}| + |\{x \in \mathbb{I} : |Sf_m(x) - Sf(x)| > y\}| \leqslant \frac{C}{y}\|f - f_m\|_1,$$

where the constant C is independent of y and m.

Since $\|f - f_m\|_1 \to 0$, hence $|\{x : \limsup_{n\to\infty} |T_n f(x) - Sf(x)| > 2y\}| = 0$ for every $y > 0$. We thus get $\limsup_{n\to\infty} |T_n f - Sf| = 0$ a.e. on $\mathbb{I}$. ■

[1] Here and in Lemma 4.3 below $L_0(\mathbb{I})$ denotes the space of measurable functions.

Below we will use the notion of quasi-local operator (see [18]):

Operator $T : L_1(\mathbb{I}) \to L_0(\mathbb{I})$ is called *quasi-local* if there is a constant C such that for every dyadic interval I and every function $f \in L_1(\mathbb{I})$ satisfying $\mathrm{supp} f \subset I$ we have

$$\int_{(2*I)^c} |Tf(x)|dx \leqslant C\|f\|_1.$$

Lemma 4.3. *If the subadditive operator $T : L_1(\mathbb{I}) \to L_0(\mathbb{I})$ is quasi-local and of the weak-type $(2,2)$, then T is also of the weak-type $(1,1)$.*

The proof is similar to the proof of Theorem 6.2.4 from [18]. ∎

The proof of Theorem 4.1: For the natural number $n > 1$ we define μ and η as the unique natural numbers such that $n = 2^\mu + \eta$ and $1 \leqslant \eta \leqslant 2^\mu$. Using this notation we can write

$$\begin{aligned}\sigma_n f &= \frac{1}{n}\left(\sum_{l=1}^{2^\mu} S_l f + \sum_{l=1}^{\eta} S_{2^\mu+l} f\right) = \\ &= \frac{1}{n}\left(S_1 f + \sum_{i=0}^{\mu-1}\sum_{l=1}^{2^i}\left(S_{2^i+l}f - S_{2^i}f\right) + \sum_{i=0}^{\mu-1} 2^i S_{2^i} f + \right. \\ &\left. + \sum_{l=1}^{\eta}\left(S_{2^\mu+l}f - S_{2^\mu}f\right) + \eta S_{2^\mu} f\right) =: \\ &=: T_n^{(1)} f + T_n^{(2)} f + T_n^{(3)} f,\end{aligned}$$

where

$$\begin{aligned}T_n^{(1)} f &= \frac{1}{n}\left(S_1 f + \sum_{i=0}^{\mu-1} 2^i S_{2^i} f + \eta S_{2^\mu} f\right) \\ T_n^{(2)} f &= \frac{1}{n}\left(\sum_{i=0}^{\mu-1}\sum_{l=1}^{2^i}\left(S_{2^i+l}f - S_{2^i}f\right)\right) \\ T_n^{(3)} f &= \frac{1}{n}\left(\sum_{l=1}^{\eta}\left(S_{2^\mu+l}f - S_{2^\mu}f\right)\right).\end{aligned}$$

Since $S_{2^i} f = P_{2^i} f$, where $P_{2^i} f$ is the partial sum of f with respect to the unbounded system $\{\psi_n\}$, we get

$$T_n^{(1)} f(x) = \frac{1}{n} P_1 f(x) + \frac{\eta}{n} P_{2^\mu} f(x) + \frac{1}{n}\sum_{i=0}^{\mu-1} 2^i P_{2^i} f(x).$$

It follows by (ii) of Theorem 2.2 that $P_{2^i} f(x) \to f(x)$ a.e. on $\mathbb{I}$. Let us fix $x \in \mathbb{I}$, for which the above convergence is true. Let us choose any $\epsilon > 0$. Then there is M such that for $i \geqslant M$ we have

$$|P_{2^i} f(x) - f(x)| < \epsilon.$$

For $n > 2^M$ we can write

$$|T_n^{(1)}f(x) - f(x)| \leqslant$$
$$\leqslant \frac{1}{n}|P_1 f(x) - f(x)| + \frac{\eta}{n}|P_{2^\mu} f(x) - f(x)| + \frac{1}{n}\sum_{i=0}^{\mu-1} 2^i |P_{2^i} f(x) - f(x)| \leqslant$$
$$\leqslant \frac{1}{n}|P_1 f(x) - f(x)| + \frac{\eta}{n}\epsilon + \frac{1}{n}\sum_{i=0}^{M-1} 2^i |P_{2^i} f(x) - f(x)| + \frac{1}{n}\sum_{i=M}^{\mu-1} 2^i \epsilon \leqslant$$
$$\leqslant \epsilon + \frac{2^M}{n} \sup_{0 \leqslant i < M} |P_{2^i} f(x) - f(x)|,$$

hence $T_n^{(1)} f(x) \to f(x)$ a.e. on $\mathbb{I}$.

It remains to prove that

$$\lim_{n\to\infty} (T_n^{(2)} f(x) + T_n^{(3)} f(x)) = 0 \quad a.e. \tag{14}$$

Since for any function $\overset{b}{\psi_i}$ we have $\sigma_n \overset{b}{\psi_i}(t) \to \overset{b}{\psi_i}(t)$, and the finite linear combinations of $\overset{b}{\psi_n}$ are dense in $L_1(\mathbb{I})$, the convergence (14) is fulfilled on the dense subset. It is not hard to prove that operators $T^{(2)}$ and $T^{(3)}$ defined by

$$T^{(m)} f := \limsup_{n\to\infty} |T_n^{(m)} f|, \quad m = 2, 3$$

are of type $(2,2)$. In fact, from the definitions of $T_n^{(2)}$ and $T_n^{(3)}$ we have

$$T^{(2)} \leqslant \tilde{T}^*, \quad T^{(3)} \leqslant \tilde{T}^*,$$

Therefore the type (2,2) follows from (10). By Lemmas 4.2 and 4.3 it suffices to show that the operators $T^{(2)}$ and $T^{(3)}$ are quasi-local.

In [20] F. Weisz gives formulae and estimates for the kernels of the operators $T_n^{(2)}$ and $T_n^{(3)}$ in the case of the bounded Ciesielski systems ([20], Theorem 1 and Lemma 1). Below we extend his result in the general version (Lemmas 4.4 and 4.5).

Lemma 4.4. *For $n = 2^\mu + \eta$ ($\mu \geqslant 0$, $1 \leqslant \eta \leqslant 2^\mu$) we have*

$$T_n^{(2)} f(x) = \frac{1}{n}\sum_{i=0}^{\mu-1} \int_0^1 L_i(x,t) f(t)\,dt,$$
$$T_n^{(3)} f(x) = \frac{1}{n}\int_0^1 M_n(x,t) f(t)\,dt,$$

where

$$L_i(x,t) = \int_0^1 \int_0^1 r_i(s \dot{+} u) 2^i F_{2^i}(s \dot{+} u) G_i^\psi(x,s) G_i^\phi(t,u)\,ds\,du,$$
$$M_n(x,t) = \int_0^1 \int_0^1 r_\mu(s \dot{+} u) \eta F_\eta(s \dot{+} u) G_\mu^\psi(x,s) G_\mu^\phi(t,u)\,ds\,du,$$

and G_n^ψ, G_n^ϕ are defined by (6) *for the systems $\{\psi_n\}$ and $\{\phi_n\}$*, respectively.

Proof. We will present the sketch of the proof only for $T_n^{(2)}$. Since

$$\overset{b}{\psi}_{2^i+k}(t) = \int_0^1 w_{2^i+k}(s) G_i^{\psi}(t,s)ds = \int_0^1 r_i(s) w_k(s) G_i^{\psi}(t,s)ds,$$
$$\overset{b}{\phi}_{2^i+k}(t) = \int_0^1 w_{2^i+k}(s) G_i^{\phi}(t,s)ds = \int_0^1 r_i(s) w_k(s) G_i^{\phi}(t,s)ds,$$

hence

$$\sum_{l=1}^{2^i}\sum_{k=1}^{l}(f, \phi_{2^i+k})\psi_{2^i+k}(x) =$$
$$= \int_0^1 \Big(\int_0^1 \int_0^1 \sum_{l=1}^{2^i}\sum_{k=1}^{l} r_i(s) w_k(s) r_i(u) w_k(u) G_i^{\phi}(t,u) G_i^{\psi}(x,s) ds du \Big) f(t)dt.$$

By the definitions of D_n and F_n we obtain

$$\sum_{l=1}^{2^i}\sum_{k=1}^{l} r_i(s)w_k(s)r_i(u)w_k(u) = r_i(s\dot{+}u)\sum_{l=1}^{2^i}\sum_{k=1}^{l} w_k(s\dot{+}u) =$$
$$= r_i(s\dot{+}u)\sum_{l=1}^{2^i} D_l(s\dot{+}u) = r_i(s\dot{+}u)2^i F_{2^i}(s\dot{+}u).$$

Putting above formulae to the formula of $T_n^{(2)}$ we obtain the thesis. ■

Lemma 4.5. *Using notation of Lemma 4.4 we have*

$$|L_i(x,t)| \leqslant C2^i \sum_{j=0}^{i} 2^j \Big(S(2^{i-1}d(x, \tau_{\frac{1}{2^{j+1}}}t)) + S(2^{i-1}d(x, \tau_{-\frac{1}{2^{j+1}}}t)) \Big) \tag{15}$$

and

$$|M_n(x,t)| \leqslant$$
$$\leqslant C \sum_{i=0}^{\mu-1} 2^i \sum_{j=0}^{i} 2^j \sum_{l=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} \Big(S(2^{\mu-1}d(x, \tau_{\frac{l}{2^{\mu}}}t)) + S(2^{\mu-1}d(x, \tau_{-\frac{l}{2^{\mu}}}t)) \Big) \tag{16}$$

where

$$\tau_h t = \begin{cases} t+h & \text{for } 0 \leqslant t < 1-h \\ t+h-1 & \text{for } 1-h \leqslant t < 1. \end{cases}$$

Proof. Since the proof of Lemma 4.5 is similar to the proof of the original lemma of Weisz from [20], we present only the sketch for L_i.

By the definition of L_i, G_i and by (12) we conclude that

$$|L_i(x,t)| \leqslant$$
$$\leqslant C2^i \sum_{j=0}^{i} 2^j \sum_{k=1}^{2^i} \sum_{l=1}^{2^i} \int_{I_{i,k}} \int_{I_{i,l}} D_{2^i}(s+u+\frac{1}{2^{j+1}}) |\psi_{2^i+k}(x)||\phi_{2^i+l}(t)| ds du \leqslant$$
$$\leqslant C2^i \sum_{j=0}^{i} 2^j \sum_{k=1}^{2^i} \sum_{l=1}^{2^i} \int_{I_{i,k}} \int_{I_{i,l}} D_{2^i}(s+u+\frac{1}{2^{j+1}}) 2^i S(2^i d(x,\frac{k}{2^i})) S(2^i d(t,\frac{l}{2^i})).$$

From (11) it follows that if we fix j and k then there are at most two numbers l such that the integral $\int_{I_{i,k}} \int_{I_{i,l}} D_{2^i}(s+u+\frac{1}{2^{j+1}}) ds du$ is not equal to zero. For $j = i, i-1$ we have than $l = k$, and for $j = 0, ..., i-2$ we have $l = k \pm 2^{i-j-1}$. Hence

$$|L_i(x,t)| \leqslant$$
$$\leqslant C2^i \sum_{j=0}^{i} 2^j \sum_{k=1}^{2^i} S(2^i d(x,\frac{k}{2^i})) \left[S(2^i d(t,\frac{k+2^{i-j-1}}{2^i})) + S(2^i d(t,\frac{k-2^{i-j-1}}{2^i})) \right] =$$
$$= C2^i \sum_{j=0}^{i} 2^j \sum_{k=1}^{2^i} S(2^i d(x,\frac{k}{2^i})) \left[S(2^i d(\tau_{\frac{1}{2^{j+1}}} t,\frac{k}{2^i})) + S(2^i d(\tau_{-\frac{1}{2^{j+1}}},\frac{k}{2^i})) \right].$$

Hence by Lemma 2.1 we have

$$|L_i(x,t)| \leqslant C2^i \sum_{j=0}^{i} 2^j \left[S(2^{i-1} d(x,\tau_{\frac{1}{2^{j+1}}} t)) + S(2^{i-1} d(x,\tau_{-\frac{1}{2^{j+1}}} t)) \right].$$

The estimates for M_n we obtained similarly using (13). ∎

Now we are ready to prove that $T^{(2)}$ and $T^{(3)}$ are quasi-local, which will complete the proof of Theorem 4.1.

Lemma 4.6. *The operator $T^{(2)} f := \limsup_{n\to\infty} |T_n^{(2)} f|$ is quasi-local.*

Proof. Let I be any dyadic interval of lenght 2^{-K}, and f let be a function from $L_1(\mathbb{I})$ with support contained in I. Then, according to the definition of $T^{(2)}$ and the assumption concerning the support of f we have

$$T^{(2)} f(x) \leqslant \limsup_{n\to\infty} |\frac{1}{n} \int_I \sum_{i=0}^{\mu-1} L_i(x,t) f(t) dt| \leqslant \sup_{n>2^K} \frac{1}{n} \sup_{t\in I} \sum_{i=0}^{\mu-1} |L_i(x,t)| \cdot \|f\|_1.$$

Hence

$$\int_{(2*I)^c} T^{(2)} f(x) dx \leqslant \int_{(2*I)^c} \sup_{n>2^K} \frac{1}{n} \sup_{t\in I} \sum_{i=0}^{\mu-1} |L_i(x,t)| dx \cdot \|f\|_1.$$

The estimate for $L_i(x,t)$ in (15) consists of two terms. We show calculations for the first one, for the second they are similar. We divide the considered integral into three pieces:

$$\int_{(2*I)^c} \sup_{n>2^K} \left(\frac{1}{n} \sum_{i=0}^{\mu-1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) \right) dx \leqslant$$

$$\leqslant \int_{(2*I)^c} \sup_{n>2^K} \left(\frac{1}{n} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) \right) dx +$$

$$+ \int_{(2*I)^c} \sup_{n>2^K} \left(\frac{1}{n} \sum_{i=K}^{\mu-1} 2^i \sum_{j=0}^{K-1} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) \right) dx +$$

$$+ \int_{(2*I)^c} \sup_{n>2^K} \left(\frac{1}{n} \sum_{i=K}^{\mu-1} 2^i \sum_{j=K}^{i} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) \right) dx =:$$

$$=: A_1 + A_2 + A_3.$$

The term A_1 is estimated as follows:

$$A_1 = \int_{(2*I)^c} \sup_{n>2^K} \left(\frac{1}{n} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) \right) dx \leqslant$$

$$\leqslant \sum_{l=1}^{2^K} \int_{I_{K,l}} \frac{1}{2^K} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) dx \leqslant$$

$$\leqslant \sum_{l=1}^{2^K} \frac{1}{2^K} \sup_{x\in I_{K,l}} \frac{1}{2^K} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{1}{2^j+1}} t)) \leqslant$$

$$\leqslant \frac{1}{(2^K)^2} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sum_{l=1}^{2^K} \sup_{x\in I_{K,l}} \sup_{t\in I} S(2^{i-1} d(x, \tau_{\frac{2^{K-j-1}}{2^K}} t)) \leqslant$$

$$\leqslant \frac{1}{(2^K)^2} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sum_{l=1}^{2^K} S(2^{i-1} d(I_{K,l}, \tau_{\frac{2^{K-j-1}}{2^K}} I)).$$

Note that $d(I_{K,l}, \tau_{\frac{2^{K-j-1}}{2^K}} I)$ takes only the values $\frac{m}{2^K}$, $m \in \mathbb{N}$, because $2^{K-j-1} \in \in \mathbb{N}$. Moreover, if j is fixed, then for each $m \in \mathbb{N}$ there are at most three l such that

$$d(I_{K,l}, \tau_{\frac{2^{K-j-1}}{2^K}} I) = \frac{m}{2^K}.$$

Hence

$$A_1 \leqslant \frac{C}{(2^K)^2} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sum_{m=0}^{2^K-1} S(2^{i-1} \frac{m}{2^K}).$$

Since $h \leqslant 2^{i-1}\frac{m}{2^K} < h+1$ if and only if $2^{K-i-1}h \leqslant m < 2^{K-i-1}(h+1)$, it follows by Lemma 2.1

$$A_1 \leqslant \frac{C}{(2^K)^2}\sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sum_{h=0}^{2^i} 2^{K-i}S(h) \leqslant \frac{C}{2^K}\sum_{i=0}^{K-1}\sum_{j=0}^{i} 2^j \leqslant C. \tag{17}$$

Let us estimate the second term, i.e. A_2. Since $i \geqslant K$, and the function S is nonincreasing,

$$\begin{aligned} A_2 &= \int_{(2*I)^c} \sup_{n>2^K}\left(\frac{1}{n}\sum_{i=K}^{\mu-1} 2^i \sum_{j=0}^{K-1} 2^j \sup_{t\in I} S(2^{i-1}d(x,\tau_{\frac{1}{2^{j+1}}}t))\right)dx \leqslant \\ &\leqslant \int_{(2*I)^c} \sup_{n>2^K}\left(\frac{1}{n}\sum_{i=K}^{\mu-1} 2^i \sum_{j=0}^{K-1} 2^j \sup_{t\in I} S(2^{K-1}d(x,\tau_{\frac{1}{2^{j+1}}}t))\right)dx \leqslant \\ &\leqslant \int_{(2*I)^c}\left(\sup_{n>2^K}\frac{1}{n}\sum_{i=K}^{\mu-1} 2^i\right)\left(\sum_{j=0}^{K-1} 2^j \sup_{t\in I} S(2^{K-1}d(x,\tau_{\frac{1}{2^{j+1}}}t))\right)dx \leqslant \\ &\leqslant \sum_{j=0}^{K-1} 2^j \int_{(2*I)^c} \sup_{t\in I} S(2^{K-1}d(x,\tau_{\frac{1}{2^{j+1}}}t))dx. \end{aligned}$$

Similarly as for A_1 we get

$$A_2 \leqslant \sum_{j=0}^{K-1} 2^j \sum_{l=1}^{2^K} \frac{1}{2^K} S(2^{K-1}d(I_{K,l},\tau_{\frac{2^{K-j-1}}{2^K}}I))$$

and using the same argument we obtain

$$A_2 \leqslant \frac{C}{2^K}\sum_{j=0}^{K-1} 2^j \sum_{m=0}^{2^K} S(m) \leqslant \frac{C}{2^K}\sum_{j=0}^{K-1} 2^j \leqslant C. \tag{18}$$

To estimate A_3 let us note that if $j \geqslant K$, then $\frac{1}{2^{j+1}} \leqslant \frac{1}{2}|I|$. Thus, if $d(x,I) \geqslant \frac{l}{2^K}$, then $d(x,\tau_{\frac{1}{2^{j+1}}}I) \geqslant \frac{l}{2^{K+1}}$, and if we denote $(2*I)^c = \{x:\ d(x,I) \geqslant \geqslant \frac{2}{2^K}\}$ we obtain

$$\begin{aligned} A_3 &= \int_{(2*I)^c} \sup_{n>2^K}\left(\frac{1}{n}\sum_{i=K}^{\mu-1} 2^i \sum_{j=K}^{i} 2^j \sup_{t\in I} S(2^{i-1}d(x,\tau_{\frac{1}{2^{j+1}}}t))\right)dx \leqslant \\ &\leqslant 2\sum_{l=2}^{2^K}\frac{1}{2^K}\sup_{n>2^K}\left(\frac{1}{n}\sum_{i=K}^{\mu-1} 2^i \sum_{j=K}^{i} 2^j S(2^{i-1}\frac{l}{2^{K+1}})\right) \leqslant \\ &\leqslant \sum_{l=2}^{2^K}\frac{1}{2^K}\sum_{i=K}^{\infty} 2^i S(2^{i-2}\frac{l}{2^K}) \end{aligned}$$

Thus, by Lemma 2.1 we have

$$A_3 \leqslant C\sum_{i=0}^{\infty} 2^i \sum_{l=1}^{\infty} S(2^{i-2}l) \leqslant C. \tag{19}$$

It follows from (17), (18) and (19) that

$$\int_{(2*I)^c} T^{(2)}f(x)dx \leqslant C\|f\|_1,$$

which completes the proof of Lemma 4.6. ■

Lemma 4.7. *The operator* $T^{(3)}f := \limsup_{n\to\infty} |T_n^{(3)}f|$ *is quasi-local.*

Proof. Let I denote a dyadic interval of lenght 2^{-K}, and let $f \in L_1(\mathbb{I})$ have support contained in I. Fix M such that $\log(x+1)S(x) < \frac{1}{2^K}$ for $x > 2^{M-2}$ (the existence of such M follows from (3)). Analogously, as in the proof of Lemma 4.6, we can write

$$\int_{(2*I)^c} T^{(3)}f(x)dx \leqslant \int_{(2*I)^c} \sup_{n>2^{K+M}} \frac{1}{n} \sup_{t\in I} |M_n(t,x)|dx \cdot \|f\|_1.$$

This time we present the calculations only for the first element in (16). We again decompose the integal

$$\int_{(2*I)^c} \sup_{n>2^{K+M}} \left(\frac{1}{n} \sum_{i=0}^{\mu-1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} \sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} S(2^{\mu-1}d(x,\tau_{\frac{m}{2^\mu}}t)) \right) dx$$

into three pieces and each of them we estimate separately. So

$$B_1 = \int_{(2*I)^c} \sup_{n>2^{K+M}} \left(\frac{1}{n} \sum_{i=0}^{K+1} 2^i \sum_{j=0}^{i} 2^j \sup_{t\in I} \sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} S(2^{\mu-1}d(x,\tau_{\frac{m}{2^\mu}}t)) \right) dx.$$

Since $2^{\mu-1} > 2^K$, we have from the fact that S is noincreasing

$$\sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} S(2^{\mu-1}d(x,\tau_{\frac{m}{2^\mu}}t)) \leqslant \sum_{m=1}^{2^{\mu-i+1}} S(2^K d(x,\tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{m}{2^\mu}}t)) \leqslant$$

$$\leqslant \sum_{s=0}^{2^{K-i+2}} \sum_{w=1}^{2^{\mu-K-1}} S(2^K d(I_{K,l},\tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{s}{2^{K+1}}+\frac{w}{2^\mu}}I)).$$

Let now

$$r(l,i,j,s) := \sup\{r\in\mathbb{N}:\ d(I_{K,l},\tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{s}{2^{K+1}}}I) \geqslant \frac{r}{2^K}\}.$$

Since $\frac{w}{2^\mu} \leqslant \frac{1}{2^{K+1}}$, thus if $d(I_{K,l}, \tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{s}{2^{K+1}}} I) \geqslant \frac{r}{2^K}$ for certain $r \in \mathbb{N}$, then

$$d(I_{K,l}, \tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{s}{2^{K+1}}+\frac{w}{2^\mu}} I) \geqslant \frac{r}{2^{K+1}},$$

hence

$$B_1 \leqslant \frac{C}{2^K}\sum_{l=1}^{2^K} \sup_{n>2^{K+M}} \left(\frac{1}{n}\sum_{i=0}^{K+1} 2^i \sum_{j=0}^{i} 2^j \sum_{s=0}^{2^{K-i+2}} 2^{\mu-K-1} S(r(l,i,j,s))\right) \leqslant$$
$$\leqslant \frac{C}{(2^K)^2}\sum_{l=1}^{2^K}\sum_{i=0}^{K+1} 2^i \sum_{j=0}^{i} 2^j \sum_{s=0}^{2^{K-i+2}} S(r(l,i,j,s)).$$

Changing the order of the summation we get

$$B_1 \leqslant \frac{C}{(2^K)^2}\sum_{i=0}^{K+1} 2^i \sum_{j=0}^{i} 2^j \sum_{s=0}^{2^{K-i+2}} \sum_{l=1}^{2^K} S(r(x,i,j,s)).$$

Let us note that $|\tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{s}{2^{K+1}}} I| = \frac{1}{2^K}$ therefore, for fixed i, j, s and $r \in \mathbb{N}$ there are at most four l such that $r(l,i,j,s) = r$. Thus

$$B_1 \leqslant \frac{C}{(2^K)^2}\sum_{i=0}^{K+1} 2^i \sum_{j=0}^{i} 2^j \sum_{s=0}^{2^{K-i+2}} \sum_{r=0}^{\infty} S(r) \leqslant \frac{C}{2^K}\sum_{i=0}^{K+1}\sum_{j=0}^{i} 2^j \leqslant C. \tag{20}$$

The second integral is treated similarly

$$B_2 =$$
$$= \int_{(2*I)^c} \sup_{n>2^{K+M}} \left(\frac{1}{n}\sum_{i=K+2}^{\mu-1} 2^i \sum_{j=0}^{K+1} 2^j \sup_{t\in I} \sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} S(2^{\mu-1} d(x, \tau_{\frac{m}{2^\mu}} t))\right) dx \leqslant$$
$$\leqslant C\sum_{l=1}^{2^K} \frac{1}{2^K} \sup_{n>2^{K+M}} \left(\frac{1}{n}\sum_{i=K+2}^{\mu-1} 2^i \sum_{j=0}^{K+1} 2^j \sum_{m=1}^{2^{\mu-i+1}} S(2^{\mu-1} d(I_{K,l}, \tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{m}{2^\mu}} I))\right).$$

Let us note that $|\frac{1}{2^i} - \frac{m}{2^\mu}| < \frac{1}{2^{K+1}}$ hence if for any $r \in \mathbb{N}_+$

$$d(I_{K,l}, \tau_{\frac{1}{2^{j+1}}} I) \geqslant \frac{r}{2^K},$$

then

$$d(I_{K,l}, \tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{m}{2^\mu}} I) \geqslant \frac{r}{2^{K+1}},$$

which gives

$$\sum_{m=1}^{2^{\mu-i+1}} S(2^{\mu-1}d(I_{K,l},\tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{m}{2^\mu}}I)) \leqslant 2^{\mu-i+1}S(2^{\mu-K-2}r).$$

On the other hand if $d(I_{K,l},\tau_{\frac{1}{2^{j+1}}}I) < \frac{1}{2^K}$, then

$$\sum_{m=1}^{2^{\mu-i+1}} S(2^{\mu-1}d(I_{K,l},\tau_{\frac{1}{2^{j+1}}-\frac{1}{2^i}+\frac{m}{2^\mu}}I)) \leqslant 2\sum_{s=0}^{2^{\mu-i+1}-1} S(2^{\mu-1}\frac{s}{2^\mu}) \leqslant C$$

Introducing

$$\beta(l,j) = \sup\{r \in \mathbb{N} :\ d(I_{K,l},\tau_{\frac{1}{2^{j+1}}}I) \geqslant \frac{r}{2^K}\}.$$

we note that

$$\forall j,r \quad \sharp\{l :\ \beta(j,l) = r\} \leqslant 3. \tag{21}$$

Thus,

$$B_2 \leqslant \frac{C}{2^K}\sum_{j=0}^{K+1} 2^j \cdot 3\sum_{r=1}^{2^K} \sup_{n>2^{K+M}} \left(\frac{1}{n}\sum_{i=K+2}^{\mu-1} 2^i 2^{\mu-i+1}S(2^{\mu-K-2}r)\right) +$$
$$+\frac{C}{2^K}\sum_{j=0}^{K+1} 2^j \sum_{l:\beta(j,l)=0} \sup_{n>2^{K+M}} \frac{1}{n}\sum_{i=K+2}^{\mu-1} 2^i =: B_2^{(1)} + B_2^{(2)}.$$

It follows from (21) that $B_2^{(2)} \leqslant C$. Since $S(x)$ is nonincreasing function, we may estimate $B_2^{(1)}$ in the following way:

$$B_2^{(1)} \leqslant \frac{C}{2^K}\sum_{j=0}^{K+1} 2^j \sum_{r=1}^{2^K} \sup_{n>2^{K+M}} (\mu-K-2)S(2^{\mu-K-2}) =$$
$$= \frac{C}{2^K}\sum_{j=0}^{K+1} 2^j \cdot 2^K \sup_{n>2^{K+M}} (\mu-K-2)S(2^{\mu-K-2}).$$

By the choice of M we have

$$\sup_{n>2^{K+M}} (\mu-K-2)S(2^{\mu-K-2}) < \frac{C}{2^K}.$$

Consequently, we get

$$B_2^{(1)} \leqslant \frac{C}{2^K}\sum_{j=0}^{K+1} 2^j \leqslant C.$$

Finally

$$B_2 \leqslant C. \tag{22}$$

It remains to estimate the third term. We have

$$\begin{aligned}
B_3 = \\
&= \int_{(2*I)^c} \sup_{n>2^{K+M}} \left(\frac{1}{n} \sum_{i=K+2}^{\mu-1} 2^i \sum_{j=K+2}^{i} 2^j \sup_{t\in I} \sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} S(2^{\mu-1} d(x, \tau_{\frac{m}{2^\mu}} t)) \right) dx \leqslant \\
&\leqslant \sum_{l=2}^{2^K} \int_{(l+1)*I\setminus l*I} \sup_{n>2^{K+M}} \left(\frac{1}{n} \sum_{i=K+2}^{\mu-1} 2^i \sum_{j=K+2}^{i} 2^j \sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} S(2^{\mu-1} d(x, \tau_{\frac{m}{2^\mu}} I)) \right) dx \leqslant \\
&\leqslant \sum_{l=2}^{2^K} \frac{2}{2^K} \sup_{n>2^{K+4}} \left(\frac{1}{n} \sum_{i=K+2}^{\mu-1} 2^i \sum_{j=K+2}^{i} 2^j \sum_{m=2^{\mu-j-1}-2^{\mu-i}+1}^{2^{\mu-j-1}+2^{\mu-i}-1} \sup_{x\in(l+1)*I\setminus l*I} S(2^{\mu-1} d(x, \tau_{\frac{m}{2^\mu}} I)) \right).
\end{aligned}$$

Since $|\frac{m}{2^\mu}| \leqslant \frac{1}{2^{K+1}}$, thus if $d(x, I) \geqslant \frac{l}{2^K}$, then $d(x, \tau_{\frac{m}{2^\mu}} I) \geqslant \frac{l}{2^{K+1}}$. Therefore

$$\begin{aligned}
B_3 &\leqslant \frac{C}{2^K} \sum_{l=2}^{2^K} \sup_{n>2^{K+M}} \left(\frac{1}{n} \sum_{i=K+2}^{\mu-1} 2^i \sum_{j=K+2}^{i} 2^j 2^{\mu-i+1} S(2^{\mu-1} \frac{l}{2^{K+1}}) \right) \leqslant \\
&\leqslant \frac{C}{2^K} \sum_{l=2}^{2^K} \sup_{n>2^{K+M}} \left(\frac{2^\mu}{n} \sum_{i=K+2}^{\mu-1} S(2^{\mu-K-2} l) \sum_{j=K+2}^{i} 2^j \right) \leqslant \\
&\leqslant \frac{C}{2^K} \sum_{l=2}^{2^K} \sup_{n>2^{K+M}} \left(S(2^{\mu-K-2} l) 2^{\mu-1} \right) \leqslant \\
&\leqslant C \sum_{l=2}^{2^K} \sum_{j=0}^{\infty} 2^j S(2^j l).
\end{aligned}$$

Changing the order of the summation we get from Lemma 2.1

$$B_3 \leqslant C \sum_{j=0}^{\infty} 2^j \sum_{l=1}^{2^K} S(2^j l) \leqslant C. \tag{23}$$

Lemma 4.7 follows now from (20), (22) and (23). ∎

References

[1] P. Auscher, *Ondelettes à support compact et conditions aux limites*, J. Funct. Anal. 111 (1993), 29–43.

[2] P. Billard, *Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2(0,1)$*, Studia Math. 28 (1967), 363–388.

[3] S.V. Bočkariev, *Divergent Fourier series on a set of positive measure for any bounded orthonormal system*, Mat. Sbornik 98 (140) (1975), 435–449.

[4] Z. Ciesielski, *A bounded orthonormal system of polygonals*, Studia Math. 31 (1968), 339–346.

[5] Z. Ciesielski, *Equivalence, unconditionality and convergence a.e. of the spline bases in L_p spaces*, Approximation Theory, Banach Center Publications, Vol. 4, PWN, Warsaw (1979), 55-68.

[6] Z. Ciesielski, J. Domsta, *Estimates for the spline orthonormal functions and for their derivatives*, Studia Math. 44 (1972), 315–320.

[7] Z. Ciesielski, S. Kwapień, *Some properties of the Haar, Walsh-Paley, Franklin and the bounded polygonal orthonormal bases in L_p spaces*, Commentationes Mathematicae, Tomus specialis in honorem Ladislai Orlicz II, (1979), 37–42.

[8] Z. Ciesielski, P. Simon, P. Sjölin, *Equivalence of Haar and Franklin bases in L_p spaces*, Studia Math. 60 (1977), 195–210.

[9] A. Cohen, I. Daubechies, P. Vial, *Wavelets on the interval and fast wavelet transforms*, Appl. Comp. Harm. Anal. 1 (1993), 54–81.

[10] N.J. Fine, *Cesàro summability of Walsh-Fourier series*, Proc. Nat. Acad. Sci. USA 41 (1955), 588–591.

[11] G.G. Gevorkyan, B. Wolnik, *Square functions for wavelet type systems*, Bull. Pol. Ac.: Math. 46 (1998), 155–172.

[12] A. Kamont, B. Wolnik, *The Walsh transform of wavelet type systems: divergence almost everywhere*, Functiones et Approximatio XXXII (2004), 57–66.

[13] K.S. Kazarian, A.S. Sargsian, *Divergence almost everywhere of Fourier-Ciesielski system*, Izv. Akad. Nauk Armyan. SSR ser. mat. 24 no. 5 (1989), 418–427 (in Russian).

[14] S.E. Kelly, M.A. Kon, L.A. Raphael, *Pointwise convergence of wavelet expansions*, J. Funct. Anal. 126 (1994), 102–138.

[15] Y. Meyer, *Ondelettes sur l'intervalle*, Rev. Mat. Iberoamericana 7 (1991), 115–133.

[16] P. Petrushev, *Bases consisting of rational functions of uniformly bounded degrees or more general functions*, J. Funct. Anal. 174 (2000), 18–75.

[17] S. Ropela, *Properties of bounded orthonormal spline bases*, Approximation Theory, Banach Center Publications 4 (1979), 197–205.

[18] F. Schipp, W.R. Wade, P. Simon, *Walsh series; an introduction to dyadic harmonic analysis*, Akadémiai Kiadó, Budapest 1990.

[19] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Arktiv Mathematik 7 (1968), 551–570.

[20] F. Weisz, *About the Fejér means of the bounded Ciesielski systems*, Studia Math. 146 (2001), 227–243.

[21] P. Wojtaszczyk, *Wavelets as unconditional bases in $L_p(\mathbb{R})$*, J. Fourier Anal. Appl. 5 (1999), 73–85.
[22] B. Wolnik, *On some class of bases in function spaces*, Ph.D. Thesis, Faculty of Mathematics and Physics, Gdańsk University, Gdańsk 2000 (in Polish).

Address: Institute of Mathematics, Gdańsk University, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail: Barbara.Wolnik@math.univ.gda.pl
Received: 15 December 2003