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ON THE SUM OF A PRIME AND A k-FREE NUMBER

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Abstract: We prove a refined asymptotic formula for the number of representations of sufficiently large integer as a sum of a prime and a k-free number, $k \geqslant 2$.
Keywords: prime numbers, k-free numbers.

1. Introduction

The problem of counting the number of representations of an integer as a sum of a prime and a square-free integer was first considered by Estermann [3] in 1931. He obtained an asymptotic formula that was subsequently refined by Page [11] and then by Walfisz [13] in 1936. In 1949 Mirsky [10] generalized such results to the case of the sum of a prime and a k-free number, where $k \geqslant 2$ is a fixed integer. He obtained, for every $A > 0$, that

$$r_k(n) = \sum_{p \leqslant n} \mu_k(n-p) = \mathfrak{S}_k(n)\mathrm{li}(n) + O\left(\frac{n}{\log^A n}\right) \quad \text{as } n \to +\infty, \tag{1}$$

where $\mu_k(n) = \sum_{a^k | m} \mu(a)$ is the characteristic function of the k-free numbers, $\mu(n)$ is the Möbius function, $\mathrm{li}(n) = \int_2^n \frac{dt}{\log t}$ and

$$\mathfrak{S}_k(n) = \prod_{p \nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \tag{2}$$

is the singular series of this problem.

The aim of this paper is to prove a refinement of Walfisz-Mirsky asymptotic formula (1). This refinement depends on inserting a new term connected with the existence of the Siegel zero of Dirichlet L-functions (see Lemmas 1-2 below) and by sharping the error term in the asymptotic formula.

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Denoting by $\Lambda(n)$ the von Mangoldt function, we define

$$R_k(n) = \sum_{m\leqslant n} \Lambda(m)\mu_k(n-m)$$

to be the weighted number of representations of an integer n as a sum of a prime and a k-free number. As usual R_k is easily related with r_k. We have the following

Theorem. *Let $k \geqslant 2$ be a fixed integer. Then there exists a constant $c = c(k) > 0$ such that, for every sufficiently large $n \in \mathbb{N}$, we have*

$$R_k(n) = \left(n - \delta_{\widetilde{\beta}}\widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}}\right)\mathfrak{S}_k(n) + O_k(nG\exp(-c\sqrt{\log n})),$$

where $\widetilde{\beta}$ is the Siegel zero, $\widetilde{\chi}$ is the Siegel character, $\widetilde{r}$ is the Siegel modulus associated with the set of Dirichlet L-functions with modulus $q \leqslant \exp(c'\sqrt{\log n})$, where $c' = c'(k) > 0$ is a suitable constant,

$$G = \begin{cases} (1-\widetilde{\beta})\sqrt{\log n} & \text{if } \widetilde{\beta} \text{ exists} \\ 1 & \text{if } \widetilde{\beta} \text{ does not exist,} \end{cases} \qquad \delta_{\widetilde{\beta}} = \begin{cases} 1 & \text{if } \widetilde{\beta} \text{ exists} \\ 0 & \text{if } \widetilde{\beta} \text{ does not exist} \end{cases}$$

(see also Lemmas 1-2 below).

An analogous result, but with a weaker error term, can also be obtained via the circle method using some recent results on exponential sums over k-free numbers proved by Brüdern-Granville-Perelli-Vaughan-Wooley [1].

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2. Lemmas

We recall now some analytic results on the zero-free region of Dirichlet L-functions.

Lemma 1. [Davenport [2], §13-14] *Assume $T' \geqslant 0$. There exists a constant $c_1 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever*

$$\sigma \geqslant 1 - \frac{c_1}{\log T'}, \quad |t| \leqslant T'$$

for all the Dirichlet characters χ modulo $q \leqslant T'$, with the possible exception of at most one primitive character $\widetilde{\chi}\,(\mathrm{mod}\ \widetilde{r})$, $\widetilde{r} \leqslant T'$. If it exists, the character $\widetilde{\chi}$ is real and the exceptional zero $\widetilde{\beta}$ of $L(s, \widetilde{\chi})$ is unique, real, simple and there exists a constant $c_2 > 0$ such that

$$\frac{c_2}{\widetilde{r}^{1/2}\log^2\widetilde{r}} \leqslant 1 - \widetilde{\beta} \leqslant \frac{c_1}{\log T'}, \quad |t| \leqslant T'.$$

Fix now $T_1 > 0$ such that $\log T_1 \asymp \sqrt{\log n}$. According to Lemma 1, applied with $T' = T_1$, we denote by $\widetilde{\beta}$ the Siegel zero, $\widetilde{\chi}$ the Siegel character and by $\widetilde{r}$ its modulus. Let now

$$T_2 = \begin{cases} T_1 & \text{if } \tilde{r} \leqslant T_1^{1/4} \\ T_1^{1/4} & \text{otherwise.} \end{cases}$$

Now Lemma 1 remains true for $T' = T_2$, with a suitable change in the constant c_1. In the following we will continue to call c_1 this modified constant. Hence $\widetilde{r} \leqslant T_2^{1/4}$, if it exists. From now on we set $T = T_2$.

Moreover we need also the following form of Deuring-Heilbronn phenomenon whose proof can be found in Knapowski [9], see also §4 of Gallagher [5].

Lemma 2. *Under the same hypotheses of Lemma 1 applied with $T' = T$, if $\widetilde{\beta}$ exists, then for all the Dirichlet characters χ modulo $q \leqslant T$, there exists a constant $c_3 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever*

$$\sigma \geqslant 1 - \frac{c_3}{\log T} \log\left(\frac{ec_1}{(1-\widetilde{\beta})\log T}\right), \quad |t| \leqslant T$$

and $\widetilde{\beta}$ is still the only exception.

The next Lemma is the explicit formula for $\psi(x, \chi)$.

Lemma 3. [Davenport [2], §19] *Let χ a Dirichlet character to the modulus q and $2 \leqslant T \leqslant x$. Then*

$$\sum_{m \leqslant x} \Lambda(m)\chi(m) = \delta_\chi x - \delta_{\chi,\widetilde{\chi}} \frac{x^{\widetilde{\beta}}}{\widetilde{\beta}} - \sum_{|\rho| \leqslant T}{}' \frac{x^\rho}{\rho} + O\left(\frac{x}{T}\log^2 qx + x^{1/4}\log x\right),$$

where $\delta_\chi = 1$ if χ is the principal character, $\delta_\chi = 0$ otherwise, $\delta_{\chi,\widetilde{\chi}} = 1$ if $\chi = \widetilde{\chi}$ and $\delta_{\chi,\widetilde{\chi}} = 0$ otherwise and $\sum'$ means that the sum runs over the non-exceptional zeros.

We will need also a zero-density result for Dirichlet's L-functions.

Lemma 4. [Huxley [7] and Ramachandra [12]] *Let χ be a Dirichlet character (mod q) and $N(\sigma, T, \chi) = |\{\rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \geqslant \sigma$ and $|\gamma| \leqslant T\}|$. Then, for $\sigma \in [1/2, 1]$, there exists a positive absolute constant c_4 such that*

$$\sum_\chi N(\sigma, T, \chi) \ll (qT)^{12/5(1-\sigma)}(\log qT)^{c_4}. \tag{3}$$

3. Proof of the theorem

Following Walfisz [13] and Mirsky [10], we have

$$
\begin{aligned}
R_k(n) &= \sum_{m\leqslant n}\Lambda(m)\sum_{d^k|(n-m)}\mu(d) = \sum_{m\leqslant n}\Lambda(m)\Big[\sum_{\substack{d^k|(n-m)\\ d\leqslant D}}\mu(d) + \sum_{\substack{d^k|(n-m)\\ d>D}}\mu(d)\Big] = \\
&= \sum_{d\leqslant D}\mu(d)\sum_{\substack{m\leqslant n\\ d^k|(n-m)}}\Lambda(m) + \sum_{d>D}\mu(d)\sum_{\substack{m\leqslant n\\ d^k|(n-m)}}\Lambda(m) = \qquad (4)\\
&= \sum_{d\leqslant D}\mu(d)\psi(n;d^k,n) + \sum_{d>D}\mu(d)\psi(n;d^k,n) = A+B,
\end{aligned}
$$

say, where $\psi(x;q,a) = \sum_{\substack{m\leqslant x\\ m\equiv a \,(\mathrm{mod}\, q)}}\Lambda(m)$ and $1\leqslant D\leqslant n^{1/k}$ will be chosen later in (12).

First of all, we estimate B. By Brun-Titchmarsh Theorem, see, *e.g.*, Friedlander-Iwaniec [4], and Theorem 328 of Hardy-Wright [6], we get

$$
B \leqslant \sum_{d>D}\psi(n;d^k,n) \ll \sum_{d>D}\frac{n}{\varphi(d^k)} \ll_k n\sum_{d>D}\frac{\log\log d}{d^k} \ll_k nD^{1-k}\log\log D. \qquad (5)
$$

Then we remark that, if $(d,n)>1$, we have $\psi(n;d^k,n)\ll_k \log^2(dn)$ and hence

$$
A = \sum_{\substack{d\leqslant D\\ (d,n)=1}}\mu(d)\psi(n;d^k,n) + O_k(D\log^2(Dn)). \qquad (6)
$$

We now insert $\psi(x;q,a) = \frac{1}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\overline{\chi}(a)\psi(x,\chi)$ in (6). Hence, by Lemma 3 and the previous remarks, we get

$$
\begin{aligned}
A &= \sum_{\substack{d\leqslant D\\ (d,n)=1}}\frac{\mu(d)}{\varphi(d^k)}\Big[n - \delta_{\widetilde{\beta}}\widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} - \sum_{\substack{\chi\,(\mathrm{mod}\,d^k)\\ \chi\neq\chi_0,\widetilde{\chi}}}\overline{\chi}(n)\sideset{}{'}\sum_{|\rho|\leqslant T}\frac{n^\rho}{\rho} + \\
&\quad + O\Big(\varphi(d^k)\big(\frac{n}{T}\log^2(d^kn) + n^{1/4}\log n\big)\Big)\Big] + O_k(D\log^2(Dn)) = \\
&= \Big(n - \delta_{\widetilde{\beta}}\widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}}\Big)\sum_{\substack{d\leqslant D\\ (d,n)=1}}\frac{\mu(d)}{\varphi(d^k)} - \sum_{\substack{d\leqslant D\\ (d,n)=1}}\frac{\mu(d)}{\varphi(d^k)}\sum_{\substack{\chi\,(\mathrm{mod}\,d^k)\\ \chi\neq\chi_0,\widetilde{\chi}}}\overline{\chi}(n)\sideset{}{'}\sum_{|\rho|\leqslant T}\frac{n^\rho}{\rho} + \\
&\quad + O\Big(\sum_{\substack{d\leqslant D\\ (d,n)=1}}\big(\frac{n}{T}\log^2(d^kn) + n^{1/4}\log n\big)\Big) + O_k(D\log^2(Dn)) = \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3, \qquad (7)
\end{aligned}
$$

say.

Evaluation of Σ_1.

To evaluate the singular series we use again Theorem 328 of Hardy-Wright [6], thus obtaining

$$\sum_{\substack{d\leqslant D\\(d,n)=1}}\frac{\mu(d)}{\varphi(d^k)}=\sum_{\substack{d=1\\(d,n)=1}}^{+\infty}\frac{\mu(d)}{\varphi(d^k)}+O\Big(\sum_{d>D}\frac{1}{\varphi(d^k)}\Big)=\mathfrak{S}_k(n)+O_k(D^{1-k}\log\log D)$$

by the Euler identity and (2). Hence we easily get

$$\Sigma_1=\left(n-\delta_{\widetilde{\beta}}\widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}}\right)\mathfrak{S}_k(n)+O_k(nD^{1-k}\log\log D). \tag{8}$$

Estimation of Σ_2.

Writing $\rho=\beta+i\gamma$ we have

$$\Sigma_2\ll\sum_{\substack{d\leqslant D\\(d,n)=1}}\frac{1}{\varphi(d^k)}\sum_{\substack{\chi\,(\mathrm{mod}\,d^k)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sum_{|\rho|\leqslant T}{}'\frac{n^\beta}{|\rho|}\leqslant\sum_{\substack{q\leqslant D^k\\(q,n)=1}}\frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sum_{|\rho|\leqslant T}{}'\frac{n^\beta}{|\rho|}. \tag{9}$$

Now, to estimate Σ_2, we first split the summation over ρ according to $0<|\rho|\leqslant 1$ and $1<|\rho|\leqslant T$. Arguing as in §20 of Davenport [2] and using Lemmas 1-2, we get

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sum_{0<|\rho|\leqslant 1}{}'\frac{n^\beta}{|\rho|}\ll n^{1-f(T)}\log^2 n, \tag{10}$$

where $f(T)=\frac{c_1}{\log T}$ if the Siegel zero does not exist or $f(T)=\frac{c_3}{\log T}\log\big(\frac{ec_1}{(1-\widetilde{\beta})\log T}\big)$ if the Siegel zero exists.

In the range $1<|\rho|\leqslant T$, we follow the line of §12 of Ivić [8]. Recalling Lemmas 1-2 and 4 and Theorem 328 of Hardy-Wright [6], we have, for $D^k\leqslant T$, that

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sum_{1<|\rho|\leqslant T}{}'\frac{n^\beta}{|\rho|}\ll(\log^{c_4+3}n)\max_{1/2\leqslant\sigma\leqslant 1-f(T)}n^\sigma\max_{1\leqslant t\leqslant T}(qt)^{12/5(1-\sigma)-1} \tag{11}$$

where $f(T)$ is as in (10).

Choosing now

$$T=D^{2k}\quad\text{and}\quad T=\exp(C\sqrt{\log n}), \tag{12}$$

where $C > 0$ is an absolute constant, we split the interval over σ in two parts: the first one is for $\sigma \in [1/2, 7/12]$ and the second one is for $\sigma \in [7/12, 1 - f(T)]$. In the first case the maxima are attained at $t = T$ and $\sigma = 7/12$ and in the second case they are attained at $t = 1$ and $\sigma = 1 - f(T)$. The total contribution of (11) is then

$$\ll (n^{7/12} + n^{1-f(T)})T^{1/2} \log^E n \ll T^{1/2} n^{1-f(T)} \log^E n, \tag{13}$$

where $E > 0$ is a suitable constant, not necessarily the same at each occurrence. An analogous argument for (10) gives the same estimate. Hence, by (10) and (12)–(13), we obtain

$$\Sigma_2 \ll T^{1/2} n^{1-f(T)} \log^E n. \tag{14}$$

If the Siegel zero does not exist than we have

$$\Sigma_2 \ll_k T^{1/2} n \exp(-c_1 \frac{\log n}{\log T}) \log^E n, \tag{15}$$

while, if the Siegel zero exists, we get

$$\begin{aligned}\Sigma_2 &\ll_k T^{1/2} n \exp\Big(- c_3 \frac{\log n}{\log T} \log \big(\frac{e c_1}{(1-\widetilde{\beta}) \log T} \big) \Big) \log^E n \ll \\ &\ll T^{1/2} n \big[(1-\widetilde{\beta}) \log T \big] \exp(-c_3 \frac{\log n}{\log T}) \log^E n,\end{aligned} \tag{16}$$

and hence, combining (15)–(16) we finally have

$$\Sigma_2 \ll T^{1/2} n G \exp(-c_5 \frac{\log n}{\log T}) \log^E n, \tag{17}$$

where $c_5 = \min(c_1; c_3)$ and

$$G = \begin{cases} (1-\widetilde{\beta})\sqrt{\log n} & \textit{if } \widetilde{\beta} \textit{ exists} \\ 1 & \textit{if } \widetilde{\beta} \textit{ does not exist.} \end{cases}$$

Estimation of Σ_3 and the final argument.

Recalling $T = D^{2k}$ and $T = \exp(C\sqrt{\log n})$, we get from (17) that

$$\Sigma_2 \ll_k nG \exp(-c_6 \sqrt{\log n}), \tag{18}$$

with

$$C = \sqrt{c_5} \quad \text{and} \quad c_6 = \sqrt{c_5}/3. \tag{19}$$

From (8) we obtain

$$\Sigma_1 = \Big(n - \delta_{\widetilde{\beta}} \widetilde{\chi}(n) \frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} \Big) \mathfrak{S}_k(n) + O_k(n \exp(-C \frac{k-1}{3k} \sqrt{\log n})). \tag{20}$$

Moreover, the error terms collected in Σ_3 can be estimated as follows:

$$\Sigma_3 \ll_k \frac{nD}{T}\log^2(D^k n) + n^{1/4}D\log n + D\log^2(Dn) \ll$$
$$\ll_k n\exp(-C\frac{2k-1}{3k}\sqrt{\log n}). \tag{21}$$

Hence, if the Siegel zero does not exist, inserting (18)–(21) into (4)–(5) and (7) we have the Theorem with $c = C\frac{k-1}{3k}$ provided that $C < \frac{3k}{k-1}c_6$ (which holds by (19)).

If the Siegel zero exists, we remark that

$$n - \widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} \geqslant n - \frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} = \int_T^n (1 - t^{\widetilde{\beta}-1})dt + O(T) \gg n(1 - T^{\widetilde{\beta}-1}) + O(T) \gg$$
$$\gg Gn + O(T)$$

and, by Lemma 1, that

$$G \gg \frac{\sqrt{\log n}}{\widetilde{r}^{1/2}\log^2\widetilde{r}} \gg \exp(-C\frac{k-1}{3k}\sqrt{\log n}),$$

since $\widetilde{r} \leqslant T^{1/4} = \exp((C/4)\sqrt{\log n})$.

Provided that $C < \frac{3k}{k-1}c_6$ (which holds by (19)), the Theorem follows also in this case with $c = C\frac{k-1}{3k}$ by inserting (18)–(21) into (4)–(5) and (7).

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